Dirac-Jacobi Bundles

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Symplectic geometry has two natural extensions:

- *presymplectic geometry*,
- *Poisson geometry*.

Dirac geometry is a common extension of both!

Remark

Mathematical Physics	Geometry
Hamiltonian mechanics (HM)	symplectic geometry
HM with constraints	presymplectic geometry
HM with symmetries	Poisson geometry
HM with both constr. and sym.	Dirac geometry

The arena for Dirac geometry is the *generalized tangent bundle*:

$$\mathbb{T}M := TM \oplus T^*M.$$

Dirac Manifolds

The main structures on $\mathbb{T}M = TM \oplus T^*M$ are:

- the projection $\mathrm{pr}_T : \mathbb{T}M \to TM$,
- the symmetric bilinear form $\langle\!\langle -, - \rangle\!\rangle : \mathbb{T}M \otimes \mathbb{T}M \to \mathbb{R}_M$:

$$\langle\!\langle (X,\sigma), (Y,\tau) \rangle\!\rangle := \tau(X) + \sigma(Y),$$

- the *Dorfman bracket* $[\![-, -]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$:

$$[\![(X,\sigma), (Y,\tau)]\!] := ([X,Y], \mathcal{L}_X \tau - i_Y d\sigma).$$

Definition

A *Dirac manifold* is a manifold M + a *Dirac structure*, i.e. a maximally isotropic subbundle $\mathfrak{D} \subset \mathbb{T}M$ such that $[\![\Gamma(\mathfrak{D}), \Gamma(\mathfrak{D})]\!] \subset \Gamma(\mathfrak{D})$.

Examples

- graphs of presymplectic forms $\omega : TM \to T^*M$,
- graphs of Poisson tensors $\pi : T^*M \to TM$,
- $T\mathcal{F} \oplus T^0\mathcal{F} \subset \mathbb{T}M$ with $\mathcal{F}$ a foliation of M.

Dirac structures are basically the same as presymplectic foliations!

Precontact and Jacobi Geometry

Contact geometry has two natural extensions:

- *precontact geometry*,
- *Jacobi geometry*.

Definition

A *precontact manifold* is a manifold + an hyperplane distribution.

Definition

A *Jacobi manifold* is a manifold M + a *Jacobi bundle*, i.e. a line bundle $L \to M$ equipped with a Lie bracket on sections

$$J : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$$

which is a 1^{st} order DO in each entry.

Every contact manifold is both a precontact and a Jacobi manifold.

Remark

There is a common extension of both precontact and Jacobi geometry.

Contact Geometry Revisited

A *contact manifold* is a manifold M + a maximally non-integrable hyperplane distribution $H \subset TM$. Dually $H = \ker(\theta : TM \to L)$.

Definition

Sections of the *Atiyah algebroid* $DE \to M$ of a vector bundle $E \to M$ are $\mathbb{R}$-linear operators $\Delta : \Gamma(E) \to \Gamma(E)$ such that

$$\Delta(fe) = (\sigma\Delta)(f)e + f\Delta(e) \quad \text{for some } \sigma\Delta \in \mathfrak{X}(M).$$

Atiyah forms are cochains in $(\Omega_E^\bullet := \Gamma(\wedge^\bullet (DE)^* \otimes E), d_{DE})$.

Proposition

Precontact structures H with $TM/H = L$ are in 1-1 correspondence with (nowhere vanishing) d_{DL}-closed Atiyah 2-forms on L. H corresponds to $\omega := d_{DL}(\theta \circ \sigma)$. H is contact iff ω in non-degenerate.

Symplectic to Contact Dictionary Principle

A contact analogue of a construction in symplectic geometry can be defined replacing the tangent bundle with the Atiyah algebroid of $L \to M$.

Dirac-Jacobi Line Bundles

The arena for Dirac-Jacobi geometry is the *omni-Lie algebroid*:

$$\mathbb{D}L := DL \oplus J^1 L \quad \text{(notice that } J^1 L = (DL)^* \otimes L\text{)}.$$

The main structures on $\mathbb{D}L$ are:

- the projection $\mathrm{pr}_D : \mathbb{D}L \to DL$,
- the symmetric bilinear form $\langle\!\langle -,- \rangle\!\rangle : \mathbb{D}L \otimes \mathbb{D}L \to L$:

$$\langle\!\langle (\Delta, \phi), (\nabla, \psi) \rangle\!\rangle := \psi(\Delta) + \phi(\nabla).$$

- the *Dorfman-Jacobi bracket* $[\![-,-]\!] : \Gamma(\mathbb{D}L) \times \Gamma(\mathbb{D}L) \to \Gamma(\mathbb{D}L)$:

$$[\![(\Delta, \phi), (\nabla, \psi)]\!] := ([\Delta, \nabla], \mathcal{L}_\Delta \psi - i_\nabla d_{DL} \phi).$$

Definition

A *Dirac-Jacobi bundle* is a line bundle $L \to M$ + a *Dirac-Jacobi structure*, i.e. a maximally isotropic subbundle $\mathfrak{D} \subset \mathbb{D}L$ such that $[\![\Gamma(\mathfrak{D}), \Gamma(\mathfrak{D})]\!] \subset \Gamma(\mathfrak{D})$.

Characteristic Foliation

Examples

- graphs of Atiyah forms $\omega : DL \to J^1L$ of precontact structures,
- graphs of Jacobi structures $J : J^1L \to DL$,
- $A \oplus A^0 \subset \mathbb{D}L$ with A a subalgebroid of DL.

Dirac-Jacobi structures are basically the same as lcps/precontact foliations!

Remark

Let $\mathfrak{D} \subset \mathbb{D}L$ be a Dirac-Jacobi structure

1. $I_{\mathfrak{D}} := \mathrm{pr}_D(\mathfrak{D})$ *is a (singular) subalgebroid of* DL,
2. $\sigma(I_{\mathfrak{D}}) = T\mathcal{F}_{\mathfrak{D}}$ *for a (singular)* characteristic foliation $\mathcal{F}_{\mathfrak{D}}$,
3. *there is a* 2*-form* $\omega_{\mathfrak{D}} : \wedge^2 I_{\mathfrak{D}} \to L$ *given by*
$$\omega_{\mathfrak{D}}(\Delta, \nabla) := \phi(\nabla), \quad \text{where } \Delta = \mathrm{pr}_D(\Delta, \phi),$$
4. $\omega_{\mathfrak{D}}$ *defines either a lcps or a precontact structure on each leaf of* $\mathcal{F}_{\mathfrak{D}}$,
5. $\mathfrak{D}$ *is completely determined by its lcps/precontact foliation.*

Jacobi Reduction

Let $\mathfrak{D} \subset \mathbb{D}L$ be a Dirac-Jacobi structure. Every leaf of $\mathcal{F}_{\mathfrak{D}}$ can be quotiented over its *null distribution* $\Rightarrow$ lcs/contact foliation $\Leftrightarrow$ a Jacobi bundle. More algebraically...

Remark (under suitable regularity conditions)

1. $\ker \omega_{\mathfrak{D}} = \mathfrak{D} \cap DL$ *and it is a subalgebroid of* DL,
2. $\sigma(\ker \omega_{\mathfrak{D}}) = T\mathcal{K}_{\mathfrak{D}}$ *for a* null foliation $\mathcal{K}_{\mathfrak{D}}$.

Definition

A section λ of L is *admissible* if $(\Delta_\lambda, j^1\lambda) \in \Gamma(\mathfrak{D})$ for some $\Delta_\lambda \in \Gamma(DL)$.

Proposition

1. *admissible sections* Γ_{adm} *form a Lie algebra under* $\{\lambda, \mu\} := \Delta_\lambda(\mu)$,
2. *a section is admissible iff it is "constant" along the leaves of* $\mathcal{K}_{\mathfrak{D}}$,
3. *there is a Jacobi bundle* $(L_{\text{red}}, J_{\text{red}})$ *over* $M/\mathcal{K}_{\mathfrak{D}}$ *and* $\Gamma(L_{\text{red}}) = \Gamma_{\text{adm}}$,
4. $\mathfrak{D}$ *is the* pull-back *of* $(L_{\text{red}}, J_{\text{red}})$.

Coisotropic Embedding

Let $(L \to M, J)$ be a Jacobi bundle.

Definition

A submanifold $S \subset M$ is *coisotropic* if sections of L vanishing on M are closed under the Jacobi bracket J.

Remark

Let $S \subset M$ be a coisotropic submanifold. (Under clean intersection*) the restricted line bundle $L|_S \to S$ carries an induced Dirac-Jacobi structure. To see this, restrict to S the lcs/contact foliation of M.*

Theorem

Let $(L \to S, \mathfrak{D})$ be a Dirac-Jacobi bundle. S can be coisotropically embedded in a manifold equipped with a Jacobi bundle iff $\operatorname{rank} \ker \omega_{\mathfrak{D}} = \text{const}$.

Proof. Work leaf-wise and use available results on coisotropic embeddings of lcps/precontact manifolds.

Let $\mathfrak{D} \subset \mathbb{D}L$ be a Dirac-Jacobi structure.

Remark

$(\mathfrak{D}, [\![-,-]\!], \sigma\,\mathrm{pr}_D)$ *is a Lie algebroid, and* L *carries a representation of* $\mathfrak{D}$.

Definition

A *precontact groupoid* is a triple $(\mathcal{G}, L, \theta)$ where

1. $\mathcal{G} \rightrightarrows M$ is a Lie groupoid with $\dim \mathcal{G} = 2\dim M + 1$,
2. $L \to M$ is a line bundle carrying a representation of $\mathcal{G}$,
3. $\theta : \mathcal{G} \to t^*L$ is a multiplicative 1-form + a *clean intersection*.

- $A \to M$ an integrable Lie algebroid,
- $\mathcal{G} \rightrightarrows M$ its source-symply connected integration,
- $L \to M$ a line bundle carrying a representation of $\mathcal{G}$ (hence of A).

Theorem

$\{$*isomorphisms* $A \simeq \mathfrak{D}\} \equiv \{$*precontact groupoid structures* (L, θ) *on* $\mathcal{G}\}$.

Credits and More Results

	$\mathcal{E}^1(M)$-Dirac	Dirac-Jacobi in $\mathbb{D}L$
definition	[Wade 2000]	[V 2015]
characteristic foliation	[Iglesias & Marrero 2002]	[V 2015]
Jacobi reduction	—	[V 2015]
coisotropic embeddings	—	[V 2015]
groupoid counterpart	[Iglesias & Wade 2006]	[V 2015]
gauge transformations	—	[V 2015]
local structure	—	[V 2015]
backward-forward maps	—	[V 2015]
Dirac-ization	[Iglesias & Marrero 2002]	[V 2015]
generalized geometry	[Iglesias & Wade 2005]	[V & Wade 2015]

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Thank you!