PROJECTABLE FORMS

Let \( M \) be an \( m \)-dimensional manifold and \( \pi: E \to M \) be a fiber bundle over \( M \) with \( \dim E = m + n \). We denote by \( X^\pi \) the \( n \)-vertical vector fields.

Definition 1. A form \( \omega \in \Omega^p(E) \) is \( \pi \)-semibasic if, and only if, \( \omega \wedge d\omega = 0 \) for every \( \omega \in \Omega^p(E) \).

Thus, to say that a form \( \omega \in \Omega^p(E) \) is \( \pi \)-basic is equivalent to say that it projects onto \( \pi^* \omega \).

HIGHER-ORDER NON-AUTONOMOUS DYNAMIC SYSTEMS

We consider a higher-order non-autonomous Lagrangian dynamical system. The configuration space is the bundle \( \pi: E \to M \), with \( \dim E = n + 1 \). The dynamical variables are the \( \pi \)-vertical vector fields \( \omega \in \Omega^p(E) \), which is the \( \pi \)-semibasic form.

Every \( \pi \)-basic form is \( \pi \)-semibasic. On the contrary, a consequence of Cartan's formula we have:

Proposition 1. A form \( \omega \in \Omega^p(E) \) is \( \pi \)-basic if, and only if, \( \omega \wedge d\omega = 0 \).

Thus, to say that a form \( \omega \in \Omega^p(E) \) is \( \pi \)-basic is equivalent to say that it projects onto \( \pi^* \omega \).

SECOND ORDER FIELD THEORIES

Now we show some consequences of the projectability of the Poincaré-Cartan form for second order Lagrangian classical field theories. The configuration bundle is \( \pi: E \to M \), with \( \dim E = n + m \). The Lagrangian form that describes the theory is a \( \pi \)-semibasic \( m \)-form \( L \in \Omega^m(E) \), and the Lagrangian function \( \mathcal{L} \in C^\infty(M) \) is such that \( \mathcal{L} = \pi^* L \), where \( \mathcal{L} \) denotes the canonical volume form in \( M \), and \( \pi^* L \) the \( \pi \)-semibasic form. The Poincaré-Cartan \( m \)-form \( \tau_{\pi^*} L \) is locally given by (see [4]):

\[
\tau_{\pi^*} L = \frac{\partial L}{\partial \pi^* (\partial^u u)} \omega(u) \wedge d\omega(u) \wedge \cdots \wedge d\omega(u) \quad (1 \leq u \leq m).
\]

where \( L \in C^\infty(E) \) are the functions given by

\[
L_j = \frac{\partial L}{\partial \pi^* (\partial^u u)} \quad (1 \leq u \leq m).
\]

Proposition 5. For \( s = 1, 2 \) the following conditions are equivalent:

1. \( \psi_{\pi^*} \) projects onto \( \pi^* \).
2. \( \psi_{\pi^*} \) is \( \pi \)-semibasic.
3. For every \( \omega \in \Omega^p(E) \), \( \pi^* \mathcal{L} = 0 \) if, and only if, \( \psi_{\pi^*} \mathcal{L} = 0 \).

(Proof) This is a consequence of a result in [5].

As in the case of higher-order dynamical systems we have the following propositions (see [5]).

Proposition 6. \( \psi_{\pi^*} \) projects onto \( \pi^* \), then the order of the Euler-Lagrange equations is at most \( s + 1 \).

Proposition 7. If there exists \( \mathcal{L} \in \Omega^m(E) \) such that \( \psi_{\pi^*} \mathcal{L} = 0 \), then \( \mathcal{L} = 0 \).

Remark: It is important to point out that, as in the case of higher-order dynamical systems, if \( \psi_{\pi^*} \) projects onto \( \pi^* \), the projected form is not necessarily the Poincaré-Cartan form for any lower order Lagrangian form \( \mathcal{L} \in \Omega^m(E) \); that is, \( \tau_{\pi^*} L \neq \mathcal{L} \).

If the Poincaré-Cartan form \( \tau_{\pi^*} L \) projects onto a lower-order jet bundle, it is associated to a highly degenerate Lagrangian (it is just a consequence of the third item in Prop. 5). Therefore:

Proposition 8. If \( \psi_{\pi^*} \) projects onto \( \pi^* \), then the solutions to the corresponding Euler-Lagrange equations exist only in the submanifold \( S \subseteq E \), where \( S \) is defined locally by the constraint functions given by

\[
l_j = 0, j = 1, \ldots, m. \quad \mathcal{L} = 0.
\]

(Proof) It is a consequence of the constraint algorithm for classical field theories described by singular Lagrangians [2] adapted to the present case. In particular, the above expressions are the compatibility conditions for the field equations. If the constraint algorithm continues by imposing the consistency conditions (tangency conditions), new constraints could appear.

EXAMPLE: HILBERT LAGRANGIAN FOR THE EINSTEIN EQUATIONS

In this example \( E = \mathbb{R}^n \) and the fiber is the space of Lorentzian metrics. The fiber coordinates are \( (x^a, u^i) \), where \( u^i \) are the components functions of the metric. The Hilbert Lagrangian function without matter is:

\[
L = \sqrt{-g} R.
\]

where \( \sqrt{-g} R \) is the scalar curvature of \( g \), which contains second-order derivatives of the components of the metric. Thus, this is a second-order field theory.

The Poincaré-Cartan form \( \tau_{\pi^*} L \) associated with the Hilbert Lagrangian density \( L = \sqrt{-g} R \) projects onto \( \pi^* L \) and hence the above propositions hold. As is well known, the corresponding Euler-Lagrange equations, which are essentially the Einstein equations, are of order 2.

Moreover, as it is noted in [5], the projected form \( \pi^* \mathcal{L} \) is not the Poincaré-Cartan form of any Lagrangian of order 1. Nevertheless, there exists a Lagrangian of order 1 whose sections solution to the corresponding Euler-Lagrange equations are equal to those of the Hilbert Lagrangian [6].

Finally, applying the Prop. 8 to the Hilbert Lagrangian, solutions to the Einstein equations exist only in a submanifold \( S \) is defined locally by the constraint functions

\[
l_j = 0 \quad \mathcal{L} = 0.
\]

The first set of conditions, evaluated on the points of the sections which are solution to the Einstein equations, are just the Einstein equations that, in this way, turn out to be constraints defining the submanifold \( S \). The second set of restrictions is related to the Bianchi identities.

References


Acknowledgements: We acknowledge the financial support of the Ministerio de Ciencia e Innovación (Spain), projects MTM2011-22885 and MTM2014-15723-E.