Geometric description of Quantum Systems in a classical-like Hamiltonian picture and applications

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Outline

**Geometric Hamiltonian formulation of QM**
- Classical tools
- Quantum Mechanics in a classical-like fashion
- Quantizer-dequantizer scheme

**Composite quantum systems**
- Liouville densities for composite quantum systems
- Separability and entanglement

**Application to quantum control**
- Quantum controllability as classical local accessibility
- Dynamical Lie algebra in terms of Killing fields
Classical tools

Phase space
A classical system with $n$ spatial degrees of freedom is described in a $2n$-dimensional symplectic manifold $(\mathcal{M}, \omega)$.

Physical state
A point $x = (q^1, ..., q^n, p_1, ..., p_n)$

Dynamics
A curve in $(a, b) \ni t \mapsto x(t) \in \mathcal{M}$ satisfying Hamilton equations:

$$\frac{dx}{dt} = X_H(x(t))$$

$H : \mathcal{M} \to \mathbb{R}$ is the Hamiltonian function.
$X_H$ is the Hamiltonian vector field, given by: $\omega_x(X_H, \cdot) = dH_x(\cdot)$
Classical tools

Statistical description

\[ \text{State of the system as a } C^1\text{-function } \rho = \rho(t, p, q). \]

Dynamics

\[ \frac{\partial \rho}{\partial t} + \{\rho, H\}_{PB} = 0 \]

Classical expecation values

Physical quantity \( f : \mathcal{M} \rightarrow \mathbb{R} \)

A state \( \rho : \mathcal{M} \rightarrow [0, 1] \) (positive, normalized to 1)

\[ \langle f \rangle_{\rho} = \int_{\mathcal{M}} f(x) \rho(t, x) d\mu(x) \]
QM in a classical-like fashion

Standard formulation of QM in a Hilbert space $\mathcal{H}$:

Quantum states: $D = \{ \sigma \in \mathfrak{B}_1(\mathcal{H}) | \sigma \geq 0, \text{tr}(\sigma) = 1 \}$
Quantum observables: Self-adjoint operators in $\mathcal{H}$.

Pure states (extremal points of $D$) are in bijective correspondence with projective rays in $\mathcal{H}$:

$$\mathcal{P}(\mathcal{H}) = \frac{\mathcal{H}}{\sim} \quad \psi \sim \phi \iff \exists \alpha \in \mathbb{C} \setminus \{0\} \text{ s.t. } \psi = \alpha \phi$$
QM in a classical-like fashion

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$\dim \mathcal{H} = n < +\infty$

$\mathcal{P}(\mathcal{H})$ is a real $(2n - 2)$-dimensional manifold with the following characterization of tangent space:

$p \in \mathcal{P}(\mathcal{H})$: $\forall \nu \in T_p \mathcal{P}(\mathcal{H}) \exists A_\nu \in i\mathfrak{u}(n)$ s.t. $\nu = -i[A_\nu, p]$.

$\mathfrak{u}(n)$ is the Lie algebra of $U(n)$.
QM in a classical-like fashion

Geometry of $\mathcal{P}(\mathcal{H})$

Symplectic form: $\omega_p(u, v) := -i k \text{tr}([A_u, A_v]p) \quad k > 0$

Riemannian metric:
$g_p(u, v) := -k \text{tr}(([A_u, p][A_v, p] + [A_v, p][A_u, p])p) \quad k > 0$

Almost complex form: $j_p : T_p \mathcal{P}(\mathcal{H}) \ni v \mapsto i[v, p] \in T_p \mathcal{P}(\mathcal{H})$ $p \mapsto j_p$ is smooth and $j_p j_p = -id$ for any $p \in \mathcal{P}(\mathcal{H})$:

$\omega_p(u, v) = g_p(u, j_p v)$

$(\mathcal{P}(\mathcal{H}), \omega, g, j)$ is a Kähler manifold.
QM in a classical-like fashion

Quantum observables as phase space functions
\[ \mathcal{O} : i\hbar(n) \ni A \mapsto f_A : \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R} \]

Quantum states as Liouville densities
\[ S : D \ni \sigma \mapsto \rho_\sigma : \mathcal{P}(\mathcal{H}) \rightarrow [0, 1] \]

Set up a Hamiltonian theory
- Equivalence Hamilton/Schrödinger dynamics:
\[ \frac{dp}{dt} = -i[H, p(t)] \Leftrightarrow \frac{dp}{dt} = X_{f_H}(p(t)) \]

- Equivalence of expectation values:
\[ \langle A \rangle_\sigma = tr(A\sigma) = \int_{\mathcal{P}(\mathcal{H})} f_A \rho_\sigma d\mu \]
From operators to functions

Let $d_g(p_1, p_2)$ be the geodesic distance w.r.t. Fubini-Study metric between $p_1$ and $p_2$. Its max is $\pi/2$.

**Definition**

$N \subset \mathcal{P}(\mathcal{H})$ is called a **basis** of $\mathcal{P}(\mathcal{H})$ if $d_g(p_1, p_2) = \frac{\pi}{2}$ for $p_1, p_2 \in \mathcal{P}(\mathcal{H})$ with $p_1 \neq p_2$ and $N$ is maximal w.r.t. this property.

**Definition**

A map $f : \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ is called **frame function** if there is $W_f \in \mathbb{C}$ s.t.

$$\sum_{p \in N} f(p) = W_f \quad \forall \text{ basis } N$$
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**Theorem (Moretti, D.P. 2013)**

$2 < \dim \mathcal{H} < \infty$. For any frame function $f$ belonging to $L^2(\mathcal{P}(\mathcal{H}), \mu)$ \exists! $A \in \mathfrak{B}(\mathcal{H})$ s.t. $f(p) = \text{tr}(Ap) \ \forall p \in \mathcal{P}(\mathcal{H})$. 
From operators to functions

\[ \mathcal{F}^2(\mathcal{H}) := \{ f : \mathcal{P}(\mathcal{H}) \to \mathbb{C} | f \in L^2(\mathcal{P}(\mathcal{H}), \mu), \ f \text{ is a frame function} \} \]

Phase space functions describing quantum observables (classical-like observable): \textit{real functions} in \( \mathcal{F}^2(\mathcal{H}) \).

Liouville probability densities describing quantum states:

\[ \left\{ \rho \in \mathcal{F}^2(\mathcal{H}) \mid \rho \geq 0, \int_{\mathcal{P}(\mathcal{H})} \rho \, d\mu = 1 \right\} \]

From operators to functions

\( \Theta : i\mathfrak{u}(n) \ni A \mapsto f_A \quad f_A(p) = k \text{tr}(Ap) + \frac{1 - k}{n} \text{tr}(A) \quad k > 0 \)

\[ S : D \ni \sigma \mapsto \rho_\sigma \quad \rho_\sigma(p) = \frac{n(n+1)}{k} \text{tr}(\sigma p) + \frac{k - (n+1)}{k} \]
C*-algebra of classical-like observables

\[ \mathcal{O} : iu(n) \ni A \mapsto f_A \] linear extension \[ \mathcal{O} : \mathcal{B}(\mathcal{H}) \to \mathcal{F}^2(\mathcal{H}) \]

\( \mathcal{F}^2(\mathcal{H}) \) as C*-algebra of observables

- Involution: \( A = \mathcal{O}(f), \ A^* = \mathcal{O}(\bar{f}); \)
- \( \ast \) - product: \( f \ast g = \mathcal{O} (\mathcal{O}^{-1}(f)\mathcal{O}^{-1}(g)) \):
  \[
  f \ast g = \frac{i}{2} \{f, g\}_{PB} + \frac{1}{2} G(df, dg) + fg \quad k = 1
  \]

- Norm: \[ |||f||| = ||\ \mathcal{O}^{-1}(f) \|| \]
  \[
  |||f||| = \frac{1}{k} \left\| f - \frac{1 - k}{n} \int_{\mathcal{P}(\mathcal{H})} f \, d\nu_n \right\|_{\infty} \quad k > 0
  \]
From functions to operators

Consider a quantum theory in geometric Hamiltonian formalism. **Observable algebra:** $\mathcal{F}^2(\mathcal{H})$.

**Coming back to standard QM**

Let $\mathcal{D} : \mathcal{P}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be defined as:

$$
\mathcal{D}(p) := \frac{n+1}{k} p - \left( \frac{n+1-k}{kn} \right) \mathbb{I}.
$$

For any classical-like observable $f : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$, the associated self-adjoint operator is given by:

$$
A = \int_{\mathcal{P}(\mathcal{H})} f(p) \mathcal{D}(p) d\nu(p).
$$

For $k = n + 1$: $A = \int f(p) p d\nu$. 
Composite quantum systems

Composite system described in $\mathcal{H}_1 \otimes \mathcal{H}_2$

The phase space is $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and not $\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$. But: $\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$ is embedded in $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by Segre embedding:

$$\text{Seg}(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|) = |\psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_2|$$

and $\text{Seg}^* (F^2(\mathcal{H}_1 \otimes \mathcal{H}_2)) = F^2(\mathcal{H}_1) \otimes F^2(\mathcal{H}_2)$. 

Measure of entanglement (D.P. 2014)

Let $\rho : \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to [0, 1]$ be a Liouville density.

$$\rho_1 := \int \mathcal{P}(\mathcal{H}_2) \text{Seg}^* \rho d\mu_2 \rho_2 := \int \mathcal{P}(\mathcal{H}_1) \text{Seg}^* \rho d\mu_1$$

$$E(\rho) = \int \mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2) \left| \text{Seg}^* \rho(p_1, p_2) - \rho_1(p_1) \rho_2(p_2) \right|^2 d\mu_1(p_1) d\mu_2(p_2)$$
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Let \( \rho : \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to [0,1] \) be a Liouville density.

\[
\rho_1 := \int_{\mathcal{P}(\mathcal{H}_2)} \text{Seg}^* \rho \ d\mu_2 \quad \rho_2 := \int_{\mathcal{P}(\mathcal{H}_1)} \text{Seg}^* \rho \ d\mu_1
\]

\[
E(\rho) = \int_{\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)} |\text{Seg}^* \rho(p_1, p_2) - \rho_1(p_1) \rho_2(p_2)|^2 d\mu_1(p_1) d\mu_2(p_2)
\]
Composite quantum systems

Liouville densities: \( S(\sigma) = \rho_\sigma : \mathcal{P}(\mathcal{H}) \to [0, 1] \) with \( \rho_\sigma(p) = tr(\sigma p) \).

Re-quantization procedure:

\[
\sigma = (n+1) \int_{\mathcal{P}(\mathcal{H})} \mathcal{S}_\mathcal{H}(p) \rho_\sigma(p) \, d\mu(p) \\
\mathcal{S}_\mathcal{H}(p) = \left( p - \frac{1}{n+1} \mathbb{I} \right)
\]
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Re-quantization procedure:

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$$S_H(p) = \left( p - \frac{1}{n+1} \mathbb{I} \right)$$

Disguised tensor product

$D(\mathcal{H}) := \{\text{Liouville densities on } \mathcal{P}(\mathcal{H})\}$

$\diamond : D(\mathcal{H}_1) \times D(\mathcal{H}_2) \to D(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\rho_{\sigma_1} \diamond \rho_{\sigma_2} = S(\sigma_1 \otimes \sigma_2)$

$$(\rho_1 \diamond \rho_2)(p) = \int_{\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)} \rho_1 \rho_2 \, tr[(S_{\mathcal{H}_1} \otimes S_{\mathcal{H}_2})p] \, d\mu_1 d\mu_2$$

The Liouville density $\rho \in D(\mathcal{H}_1 \otimes \mathcal{H}_2)$ describes a separable state if:

$$\rho = \sum_i \lambda_i \rho_1^{(i)} \diamond \rho_2^{(i)} \quad \rho_1^{(i)} \in D(\mathcal{H}_1), \rho_2^{(i)} \in D(\mathcal{H}_2)$$
Separability and entanglement

Entanglement witness
For any entangled Liouville density $\rho \in D(\mathcal{H}_1 \otimes \mathcal{H}_2)$ there is a classical-like observable $f: \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ such that:

$$\int_{\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)} f \rho d\mu < 0 \quad \text{and} \quad \int_{\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)} f \eta d\mu \geq 0$$

for any separable Liouville density $\eta$. 

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Generalized Bell’s inequality

Let $f$ be an entanglement witness and $\rho$ a Liouville density:

$$\int_{\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)} f \rho d\mu \geq 0$$

Violated by entangled densities!
Classical control

Controllability problem for a classical non-linear problem in affine form:

\[ \dot{x}(t) = X_0(t) + \sum_{i=1}^{m} X_i(x(t)) u_i(t) \]

Let \( x = x(t) \) be the solution with initial condition \( x(0) = x_0 \) and \( V \) be a neighborhood of \( x_0 \), the *reachability set* is:

\[ \mathcal{R}^V(x_0, T) := \{ \hat{x} \in \mathcal{M} | \exists u_1, \ldots, u_m \text{ s.t. } x(t) \in V, 0 \leq t \leq T, x(T) = \hat{x} \}. \]

\[ \mathcal{R}^V_T(x_0) = \bigcup_{0 < \tau \leq T} \mathcal{R}^V(x_0, \tau). \]
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\]

\[
\mathcal{R}^V_T(x_0) = \bigcup_{0 < \tau \leq T} \mathcal{R}^V(x_0, \tau). 
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\textbf{Definition}: The system is \textbf{locally accessible from} \( x_0 \in \mathcal{M} \) if \( \mathcal{R}^V_T(x_0) \) contains a non-empty open set of \( \mathcal{M} \) for all \( V \) and \( T > 0 \). If such property holds for all \( x_0 \in \mathcal{M} \) then the system is said to be \textbf{locally accessible}.
Classical vs Quantum control

Accessibility rank condition

Let \( \mathcal{C} \) be the algebra generated by \( X_0, X_1, \ldots, X_m \). The *accessibility distribution* is defined as:

\[
\mathcal{C}(x) := \text{span}\{X(x) | X \in \mathcal{C}\} \quad x \in \mathcal{M}
\]

If \( \dim \mathcal{C}(x) = n \) for every \( x \in \mathcal{M} \) then the system is locally accessible.
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Controlled $n$-level quantum system

$$i\hbar \frac{d}{dt} |\psi\rangle = \left[ H_0 + \sum_{i=1}^{m} H_i u_i(t) \right] |\psi(t)\rangle$$

Given initial condition $|\psi(0)\rangle = |\psi_0\rangle$, solution at time $t$ is

$$|\psi(t)\rangle = U(t)|\psi_0\rangle$$
Classical vs Quantum control

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Controlled $n$-level quantum system

$$i\hbar \frac{d}{dt} U(t) = \left[ H_0 + \sum_{i=1}^{m} H_i u_i(t) \right] U(t)$$

with initial condition $U(0) = \mathbb{I}$. 
Quantum controllability

Complete controllability

The $n$-level system is complete controllable if for any unitary operator $U_f \in U(n)$ there exist controls $u_1, \ldots, u_n$ and $T > 0$ such that $U(T) = U_f$.

Controllability condition on dynamical Lie algebra

The $n$-level system is completely controllable if and only if the Lie algebra generated by $\{−iH_0, \ldots, −iH_m\}$ (dynamical Lie algebra) is $\mathfrak{u}(n)$, the Lie algebra of $U(n)$.
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Geometric Hamiltonian formulation

\[
\dot{p}(t) = X_0(p(t)) + \sum_{i=1}^{m} X_i(p(t))u_i(t)
\]

$X_i$ are the Hamiltonian fields on $\mathcal{P}(\mathcal{H})$ defined by the classical-like Hamiltonians obtained with our prescription.
Some results

**Theorem (D.P. 2015)**

Consider a controlled n-level system with dynamical Lie algebra $\mathcal{L}$. Let $\mathcal{C}$ be the accessibility algebra (classical definition) of the associated classical-like system.

$$A \in \mathcal{L} \iff X_{f_{-iA}} \in \mathcal{C}$$
Some results

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Therefore we can prove:

- A quantum system is completely controllable if and only if the classical accessibility rank condition is satisfied within geometric Hamiltonian picture.

- A quantum system is completely controllable if and only if the accessibility algebra of the associated classical-like system is the Lie algebra of $g$-Killing vector fields on $\mathcal{P}(\mathcal{H})$. 
Thank you for your attention!