1. Introduction

We prove (Theorem 1) that the only natural operations between differential forms are those obtained using linear combinations, the exterior product and the exterior differential.

Our result generalises work by Palais ([4]) and Freed-Hopkins ([2]). As an application, we also deduce a theorem, originally due to Kolář ([3]), that determines those differential forms that can be associated to a connection on a principal bundle.

To be more precise, let us fix positive integers $p_1, \ldots, p_k$, and $q \geq 0$. Let us suppose that, for each manifold $X$ and each collection $\omega_1, \ldots, \omega_k$ of differential forms on $X$ of degree $p_1, \ldots, p_k$, we have defined a $q$-form $P(\omega_1, \ldots, \omega_k)$ on $X$.

Let us also assume that the assignment $(\omega_1, \ldots, \omega_k) \mapsto P(\omega_1, \ldots, \omega_k)$ is compatible with inverse images: for any smooth map $\tau: X \to X$ it holds $\tau^* [P(\omega_1, \ldots, \omega_k)] = P(\tau^* \omega_1, \ldots, \tau^* \omega_k)$.

Then, our main result (Theorem 1) proves that there exists a unique polynomial $P(x_1, y_1, \ldots, x_k, y_k)$, homogeneous of degree $q$ in anti-commutative variables of degree $\deg x_i = p_i$, $\deg y_i = p_i + 1$, such that:

$$P(x_1, \ldots, x_k) = P(y_1, \ldots, y_k),$$

for any smooth manifold $X$ and any collection $\omega_1, \ldots, \omega_k$ of differential forms on $X$ of degree $p_1, \ldots, p_k$.

In this formula, the product of variables on the polynomial $P$ has been replaced by the exterior product of forms.

The case $p_1 = \cdots = p_k = 1$ was obtained by Freed-Hopkins ([2]) using a different language and quite different methods to those applied here. When the given collection reduces to a single form $\omega$ of degree $p$, and we take $q = p + 1$, it follows that the assignment $P$ is a constant multiple of the exterior differential $P(\omega) = \lambda \omega$. This is a classical statement due to Palais ([4], Thm. 10.5), who assumed $P$ to be linear, and to Kolář-Michor-Slovak in the general case.

As an application of our results, we also prove a version of a beautiful theorem originally due to Kolář ([3]), that determines those differential forms that can be defined in a natural way from a connection on a principal bundle (Theorem 2).

In particular, it says that the Chern-Weil forms, defined by the Weil homomorphism, are the only natural differential forms that can be constructed from a $G$-connection (see also [2], Thm. 7.20).

2. Natural operations on differential forms

Let $\text{Man}$ be the category of all smooth manifolds and arbitrary smooth maps between them.

Let us fix a finite sequence of positive integers $(p_1, \ldots, p_k)$. Let us denote $\mathbb{R} \{ u_1, \ldots, u_k \}$ the anti-commutative algebra of polynomials with real coefficients in the variables $u_1, \ldots, u_k$, where each variable $u_i$ is assigned degree $p_i$.

The anti-commutative character of this algebra is expressed by the relations $u_i u_j = (-1)^{p_i p_j} u_j u_i$.

The degree of a monomial $u_1^{a_1} \cdots u_k^{a_k}$ is defined as $\sum a_i p_i$. A polynomial $P(u_1, \ldots, u_k) \in \mathbb{R} \{ u_1, \ldots, u_k \}$ is said homogeneous of degree $q$ if it is a linear combination of monomials of degree $q$.

Let $P(x_1, \ldots, x_k) \in \mathbb{R} \{ u_1, \ldots, u_k \}$ be a homogeneous polynomial of degree $q$, and let $\omega_1, \ldots, \omega_k$ be differential forms of degree $p_1, \ldots, p_k$ on a smooth manifold $X$. Then $P(\omega_1, \ldots, \omega_k)$, where the product of variables is replaced by the exterior product of forms, is a differential form of degree $q$ on $X$.

Theorem 1. Let $p_1, \ldots, p_k$ be positive integers, and let $\mathbb{R} \{ u_1, \ldots, u_k \}$ be the anti-commutative algebra of polynomials in the variables $u_1, \ldots, u_k$, of degree $\deg u_i = p_i$, $\deg v_i = p_i + 1$.

Any morphism of functors over the category $\text{Man}$

$$P: \mathbb{R}^p \oplus \cdots \oplus \mathbb{R}^p \longrightarrow \mathbb{R}^q \quad (q \geq 0)$$

can be written as

$$P(\omega_1, \ldots, \omega_k) = P(\omega_1, \omega_1, \ldots, \omega_k, \omega_k)$$

for a unique homogeneous polynomial $P(u_1, \ldots, u_k, v_1) \in \mathbb{R} \{ u_1, \ldots, u_k, v_1 \}$ of degree $q$.

3. Forms associated to a principal connection

Let us fix a Lie group $G$. Let $C$ denote the functor on $\text{Man}$ that assigns, to any smooth manifold $X$, the set of isomorphism classes of pairs $(P \to X, \alpha)$, where $P \to X$ is a principal $G$-bundle and $\alpha$ a principal connection on it.

A $q$-form naturally associated to a connection is a morphism of functors $\theta: C \rightarrow \mathbb{R}^q$.

By definition, $\theta$ assigns, to any pair $(P \to X, \alpha)$, a differential $q$-form $\tilde{\theta}(P, \alpha)$ on $X$ satisfying the following two properties:

- If $(P \to X, \alpha)$ and $(P' \to X, \alpha')$ are isomorphic, then

$$\tilde{\theta}(P, \alpha) = \tilde{\theta}(P', \alpha') \cdot \quad (\text{for any pair } (P \to X, \alpha) \text{ and any smooth map } f: Y \to X, \text{ it holds:})$$

$$\tilde{\theta}(f^* P, f^* \alpha) = f^* (\tilde{\theta}(P, \alpha)) \cdot \quad (\text{for any } \alpha \text{ and any smooth map } f: Y \to X, \text{ it holds:})$$

Theorem 2. The $2q$-forms naturally associated to a connection biunivocally correspond with the $G$-invariant linear maps $T: S^q g \rightarrow \mathbb{R}$.

By any form of odd order naturally associated to a connection is identically zero.

More precisely, the natural $2q$-form $\tilde{\theta}(P, \alpha)$ corresponding to a $G$-invariant linear map $T: S^q g \rightarrow \mathbb{R}$ is the projection on $X$ of the following $2q$-form over $P$

$$\theta(P, \alpha) := T \circ (\Theta(P, \alpha) \wedge \cdots \wedge \Theta(P, \alpha)) \cdot \quad (\text{where } \Theta \text{ is the } g \text{-valued curvature } 2\text{-form on } P)$$

This result shows that the Chern-Weil forms, defined by the Weil homomorphism, are the only natural differential forms one can construct from a $G$-connection.

References


