# On the correspondence of VB-Courant algebroids with Lie 2-algebroids

Madeleine Jotz Lean
The University of Sheffield

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# Outline

# The two vector bundle structures on TE

Take a vector bundle $q\colon E \to M$ with fiberwise addition $+$. Then its tangent bundle has two vector bundle structures:

1. The tangent bundle structure $TE \to E$;
2. The tangent prolongation $Tq\colon TE \to TM$ of $q$, with addition $T+\colon TE \times_{TM} TE \to TE$.

TE is a *double vector bundle*; i.e. the two vector bundle additions "commute".

The tangent space to the fibers of E,

$$T^qE = \ker\{Tq\colon TE \to TM\}$$

is closed under both additions.

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# The tangent space to the fibers of E



The elements of $T^qE$ can all be written

$$\left.\frac{d}{dt}\right|_{t=0} e_m + te'_m$$

for some $e_m, e'_m \in E_m$, $m \in M$.

Each section $e \in \Gamma(E)$ defines $e^\uparrow \in \Gamma(T^qE) \subseteq \mathfrak{X}(E)$ with flow

$$\Phi_t^{e^\uparrow} : E \to E, \quad e'_m \mapsto e'_m + t \cdot e(m).$$

In a similar manner, each vector bundle morphism $\phi : E \to E$ defines a vector field $\phi^\uparrow \in \Gamma(T^qE) \subseteq \mathfrak{X}(E)$ with flow

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# Usual connections and splittings of TE

$T^qE$ is so very easy to describe. A linear splitting of TE is a linear complement to $T^qE$ in TE: a complementary space

$$H \oplus T^qE \simeq TE$$

over E, that is also closed under T+.

Such linear splittings H are equivalent to linear connections $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$.

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# Usual connections and splittings of TE



Given $\nabla$, the corresponding linear splitting $H_\nabla$ is given by

$$H_\nabla(e_m) = \left\{ T_m e(v_m) - \left.\frac{d}{dt}\right|_{t=0} (e_m + t\nabla_{v_m} e) \,\middle|\, v_m \in T_m M \right\}$$

for $e_m \in E$ and any $e \in \Gamma(E)$ such that $e_m = e(m)$.

# Usual connections and splittings of TE

Alternatively, given H, there exists for each $X \in \mathfrak{X}(M)$ a unique $\widetilde{X} \in \Gamma(H)$ such that $\widetilde{X} \sim_q X$. The corresponding connection $\nabla = \nabla^H$ is then defined by

$$\mathscr{L}_{\tilde{X}}(\ell_\varepsilon) = \ell_{\nabla^*_X \varepsilon}$$

for all $\varepsilon \in \Gamma(E^*)$ and $X \in \mathfrak{X}(M)$.

# Linear connections and the tangent space of a vector bundle



To summarise, there are two equivalent definitions of a linear connection on a vector bundle $q\colon E \to M$.

$$\left\{ \begin{array}{c} \text{Linear splittings} \\ TE \cong T^qE \oplus H \text{ over } E \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{Covariant derivatives} \\ \nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \end{array} \right\}$$

▸ Splittings of $TE \oplus T^*E$

# Usual connections and the Lie bracket on $\mathfrak{X}(E)$

Then the Lie bracket of vector fields on E satisfies:

$$\left[\widetilde{X}_1, \widetilde{X}_2\right] = \widetilde{[X_1, X_2]} - R_\nabla(X_1, X_2)^\uparrow,$$
$$\left[\widetilde{X}, e^\uparrow\right] = (\nabla_X e)^\uparrow,$$

and

$$\left[e_1^\uparrow, e_2^\uparrow\right] = 0$$

for all $X, X_1, X_2 \in \mathfrak{X}(M)$ and $e, e_1, e_2 \in \Gamma(E)$.

To the Courant-Dorfman bracket.

# Usual connections and the Lie bracket on $\mathfrak{X}(E)$



This reflects the linearity of the Lie bracket of vector fields on E

$$\begin{array}{ccc} TE & \longrightarrow & E \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

# General definition of Courant algebroids



A Courant algebroid over a manifold M is a vector bundle $\mathsf{C} \to M$ with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot\,,\cdot \rangle$, a bracket $[\![\cdot\,,\cdot]\!]$ on the smooth sections $\Gamma(\mathsf{C})$, and an anchor $\rho\colon \mathsf{C} \to TM$, which satisfy the following conditions

1. $[\![c_1, [\![c_2, c_3]\!]]\!] = [\![[\![c_1, c_2]\!], c_3]\!] + [\![c_2, [\![c_1, c_3]\!]]\!]$,
2. $\rho(c_1)\langle c_2, c_3\rangle = \langle [\![c_1, c_2]\!], c_3\rangle + \langle c_2, [\![c_1, c_3]\!]\rangle$,
3. $[\![c_1, c_2]\!] + [\![c_2, c_1]\!] = \rho^* \mathbf{d}\langle c_1, c_2\rangle$

for all $c_1, c_2, c_3 \in \Gamma(\mathsf{C})$.

# The standard Courant algebroid over a smooth manifold



The best understood example of a Courant algebroid is the following.

Let M be a smooth manifold. Then the direct sum $TM \oplus T^*M$ over M, with

1. the anchor $\rho := \mathrm{pr}_{TM} : TM \oplus T^*M \to TM$,
2. the pairing

$$\langle (X_1, \theta_1), (X_2, \theta_2) \rangle = \theta_2(X_1) + \theta_1(X_2),$$

3. and the bracket

$$[\![ (X_1, \theta_1), (X_2, \theta_2) ]\!] = ([X_1, X_2], \mathscr{L}_{X_1}\theta_2 - \mathbf{i}_{X_2}\mathbf{d}\theta_1)$$

is a Courant algebroid.

# The standard Courant algebroid over a smooth manifold



A *Dirac structure* on M is a subbundle $D \subseteq TM \oplus T^*M$ with $D = D^\perp$ relative to the pairing and $[\![\Gamma(D), \Gamma(D)]\!] \subseteq \Gamma(D)$.

In particular, given a vector bundle $q\colon E \to M$, the space $TE \oplus T^*E \to E$ carries the standard Courant algebroid structure.

In fact, this Courant algebroid structure is *linear*. The space $TE \oplus T^*E$ is a double vector bundle

$$\begin{array}{ccc} TE \oplus T^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ TM \oplus E^* & \longrightarrow & M \end{array}$$

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# Original motivation

Can we split the standard Courant algebroid over a vector bundle?

# Dorfman connections and....

If TE splits as $T^qE \oplus H_\nabla$ then $TE \oplus T^*E$ splits as

$$(T^qE \oplus (T^qE)^{\mathrm{ann}}) \oplus (H_\nabla \oplus H_\nabla^{\mathrm{ann}}).$$

But this is not general enough: a general splitting of

$$0 \to T^qE \oplus (T^qE)^{\mathrm{ann}} \hookrightarrow TE \oplus T^*E \to q^!(TM \oplus E^*) \to 0$$

is not necessarily of the form $H_\nabla \oplus H_\nabla^{\mathrm{ann}}$!

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# ... linear plittings of $TE \oplus T^*E$

**Theorem (J.L. 2012)**

*There is a bijection*

$$\left\{ \begin{array}{c} \textit{Linear splittings} \\ TE \oplus T^*E \cong (T^qE \oplus (T^qE)^{\mathrm{ann}}) \oplus L \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \textit{Dorfman connections} \\ \Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M) \\ \textit{with } TM \oplus E^* \textit{ anchored by } \mathrm{pr}_{TM}. \end{array} \right\}$$

▸ Splittings of TE ▸ Theorem with dull algebroids.

# Dorfman connections



Let $Q \to M$ be a vector bundle, anchored by $\rho_Q \colon Q \to TM$. A **Dorfman (Q-)connection on $Q^*$** is an ($\mathbb{R}$-bilinear) map

$$\Delta \colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$$

satisfying

1. $\Delta_q(f\tau) = f \cdot \Delta_q\tau + \rho_Q(q)(f) \cdot \tau$,
2. $\Delta_{fq}\tau = f \cdot \Delta_q\tau + \langle q, \tau\rangle \cdot \rho_Q^*\mathbf{d}f$ and
3. $\Delta_q(\rho_Q^*\mathbf{d}f) = \rho_Q^*\mathbf{d}(\rho_Q(q)(f))$

for $q \in \Gamma(Q)$, $\tau \in \Gamma(Q^*)$ and $f \in C^\infty(M)$.

# Dull algebroids



The dual map to $\Delta$ in the sense of derivations,

$$[\![\cdot\,,\cdot]\!]_\Delta : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q),$$

$$\langle [\![q_1, q_2]\!]_\Delta, \tau \rangle = \rho_Q(q_1)\langle q_2, \tau\rangle - \langle q_2, \Delta_{q_1}\tau\rangle,$$

for $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ satisfies

$$\rho_Q [\![q_1, q_2]\!]_\Delta = [\rho_Q(q_1), \rho_Q(q_2)]$$

and (the Leibniz identities)

$$[\![q_1, f \cdot q_2]\!]_\Delta = f \cdot [\![q_1, q_2]\!]_\Delta + \rho_Q(q_1)(f) \cdot q_2$$

$$[\![f \cdot q_1, q_2]\!]_\Delta = f \cdot [\![q_1, q_2]\!]_\Delta - \rho_Q(q_2)(f) \cdot q_1$$

for all $q_1, q_2 \in \Gamma(Q)$ and $f_1, f_2 \in C^\infty(M)$.

# Dull algebroids



We call this type of bracket a *dull bracket*.

NB: We have no skew-symmetry or Jacobi identity!

But the Jacobiator in Leibniz form

$$\mathrm{Jac}_\Delta(q_1, q_2, q_3) = [\![ [\![ q_1, q_2 ]\!]_\Delta, q_3 ]\!]_\Delta + [\![ q_2, [\![ q_1, q_3 ]\!]_\Delta ]\!]_\Delta - [\![ q_1, [\![ q_2, q_3 ]\!]_\Delta ]\!]_\Delta$$

is equivalent to the *curvature* of the Dorfman connection:

$$\mathrm{Jac}_\Delta(q_1, q_2, q_3) = R_\Delta(q_1, q_2)^* q_3$$

for all $q_1, q_2, q_3$.

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$$\mathrm{Jac}_\Delta(q_1,q_2,q_3) = \mathrm{R}_\Delta(q_1,q_2)^* q_3$$

for all $q_1, q_2, q_3$.

# Equivalently

**Theorem (J.L. 2012)**

*There is a bijection*

$$\left\{ \begin{array}{c} \textit{Linear splittings} \\ TE \oplus T^*E \cong (T^qE \oplus (T^qE)^{\mathrm{ann}}) \oplus L \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \textit{Dull brackets on sections of the} \\ \mathrm{pr}_{TM}\textit{-anchored vector bundle } TM \oplus E^*. \end{array} \right\}$$

▸ Theorem with Dorfman connections.

# Courant algebroid structure of $TE \oplus T^*E \to E$



1. Sections of $E \oplus T^*M$ now lift vertically to sections of $T^qE \oplus (T^qE)^{ann} \subseteq TE \oplus T^*E$ over E: Given $\tau = (e, \theta) \in \Gamma(E \oplus T^*M)$,

$$\tau^\uparrow = \left(e^\uparrow, q^*\theta\right).$$

2. Sections of $TM \oplus E^*$ lift horizontally to sections of $L \subseteq TE \oplus T^*E$: Given $\upsilon = (X, \varepsilon) \in \Gamma(TM \oplus E^*)$, the corresponding horizontal section is given by

$$\tilde{\upsilon}(e_m) = \left(T_m eX(m), \mathbf{d}_{e_m}\ell_\varepsilon\right) - \Delta_\upsilon(e, 0)^\uparrow(e_m).$$

# The linear connection associated to a Dorfman connection

Given a Dorfman connection
$\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$,

$$\nabla\colon \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E) \qquad \nabla_v e = \mathrm{pr}_E(\Delta_v(e, 0))$$

defines a linear connection.

# Courant algebroid structure of $TE \oplus T^*E \to E$

**Theorem (J. 2012)**

*The anchor* $\rho\colon TE \oplus T^*E \to TE$ *sends*

1. *for* $v \in \Gamma(TM \oplus E^*)$*,* $\tilde{v}$ *to* $\widetilde{\nabla_v} \in \mathfrak{X}(E)$*;*

$$\mathcal{L}_{\widetilde{\nabla_v}}(q^* f) = q^*(\mathcal{L}_{pr_{TM} v}(f)) \quad \textit{and} \quad \mathcal{L}_{\widetilde{\nabla_v}}(\ell_\varepsilon) = \ell_{\nabla^*_v \varepsilon},$$

2. *and for* $\tau \in \Gamma(E \oplus T^*M)$*,* $\tau^\uparrow$ *to* $(pr_E \tau)^\uparrow$*.*

# Courant algebroid structure of $TE \oplus T^*E \to E$

## Theorem (J. 2012)

*The canonical pairing on* $TE \oplus T^*E \to E$ *satisfies:*

1. $\langle \widetilde{v_1}, \widetilde{v_2} \rangle = \ell_{[\![v_1,v_2]\!]_\Delta + [\![v_2,v_1]\!]_\Delta}$,
2. $\left\langle \tilde{v}, \tau^\uparrow \right\rangle = q^*\langle v, \tau \rangle$,
3. $\left\langle \tau_1^\uparrow, \tau_2^\uparrow \right\rangle = 0$

*for* $v, v_1, v_2 \in \Gamma(TM \oplus E^*)$ *and* $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$.

# Courant algebroid structure of $TE \oplus T^*E \to E$



**Theorem (J. 2012)**

*The Courant-Dorfman bracket on sections of* $TE \oplus T^*E \to E$ *satisfies:*

1. $[\![\widetilde{\upsilon_1}, \widetilde{\upsilon_2}]\!] = \widetilde{[\![\upsilon_1, \upsilon_2]\!]_\Delta} - R_\Delta(\upsilon_1, \upsilon_2)(\cdot, 0)^\uparrow.$
2. $\left[\!\left[\tilde{\upsilon}, \tau^\uparrow\right]\!\right] = \left(\Delta_\upsilon \tau\right)^\uparrow,$
3. $\left[\!\left[\tau_1^\uparrow, \tau_2^\uparrow\right]\!\right] = 0$

*for* $\upsilon, \upsilon_1, \upsilon_2 \in \Gamma(TM \oplus E^*)$ *and* $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$.

To the Lie bracket of vector fields.

**Theorem (Li-Bland 2012, J.L.2015)**

*The category of Lie 2-algebroids is equivalent to the category of VB-Courant algebroids.*

Splittings of a Lie 2-algebroid correspond to maximally isotropic linear splittings of the corresponding VB-Courant algebroid.

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & B \\ \downarrow & & \downarrow \\ Q & \longrightarrow & M \end{array}$$

# Split Lie 2-algebroids



A split Lie 2-algebroid $Q \oplus B^* \to M$ is a pair of an anchored vector bundle $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with

1. a vector bundle map $\partial_B^* : B^* \to Q$,
2. a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!] : \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$,
3. a linear connection $\nabla^* : \Gamma(Q) \times \Gamma(B^*) \to \Gamma(B^*)$ and
4. a vector valued 3-form $\omega \in \Omega^3(Q, B^*)$,

such that

(i) $\nabla^*_{\partial_B^*(\beta_1)}\beta_2 + \nabla^*_{\partial_B^*(\beta_2)}\beta_1 = 0$,

(ii) $[\![q, \partial_B^*(\beta)]\!] = \partial_B^*(\nabla^*_q\beta)$,

(iii) $\mathrm{Jac}_{[\![\cdot,\cdot]\!]} = -\partial_B^* \circ \omega \in \Omega^3(Q, Q)$,

(iv) $R_{\nabla^*}(q_1, q_2)\beta = \omega(q_1, q_2, \partial_B^*(\beta))$, and

(v) $\mathbf{d}_{\nabla^*}\omega = 0$

for all $\beta, \beta_1, \beta_2 \in \Gamma(B^*)$ and $q, q_1, q_2 \in \Gamma(Q)$.

Let E be a vector bundle over M, choose a *skew-symmetric* dull bracket $[\![\cdot\,,\cdot]\!]$ on $TM \oplus E^*$ and let $\Delta$ be the dual Dorfman connection.

Then $\mathrm{pr}_E^*\colon E^* \hookrightarrow TM \oplus E^*$, $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$, $[\![\cdot\,,\cdot]\!]$ and

$$\omega \in \Omega^3(TM \oplus E^*, E^*), \quad \omega(v_1, v_2, v_3) = R_\Delta(v_1, v_2)^* v_3,$$

defines a split Lie-2-algebroid $(TM \oplus E^*) \oplus E^*$.

Let E be a vector bundle over M, choose a *skew-symmetric* dull bracket $[\![\cdot\,,\cdot]\!]$ on $TM \oplus E^*$ and let $\Delta$ be the dual Dorfman connection.

Then $pr_E^*\colon E^* \hookrightarrow TM \oplus E^*$, $\nabla = pr_E \circ \Delta \circ \iota_E$, $[\![\cdot\,,\cdot]\!]$ and

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defines a split Lie-2-algebroid $(TM \oplus E^*) \oplus E^*$.

# One further application

Consider a double Lie algebroid

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & M \end{array} .$$

# One further application



The direct sum

$$\begin{array}{ccc} D_A^* \oplus D_B^* & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A \oplus B & \longrightarrow & M \end{array}$$

is a VB-Courant algebroid.

# One further application



A linear splitting of $D$ gives rise to two 2-representations which form a *matched pair* (Gracia-Saz, J.L., Mackenzie, Mehta).

A linear splitting of $D$ naturally induces a maximally isotropic decomposition of $D_A^* \oplus D_B^*$.

# One further application

A linear splitting of D gives rise to two 2-representations which form a *matched pair* (Gracia-Saz, J.L., Mackenzie, Mehta).

A linear splitting of D naturally induces a maximally isotropic decomposition of $D_A^* \oplus D_B^*$.

# One further application

The corresponding split Lie 2-algebroid is the bicrossproduct $(A \oplus B) \oplus C$ of the matched pair of 2-representations!

# In particular...

Take a Lie bialgebroid $(A, A^*)$ over a manifold $M$. Then we have two notions of *double* of the Lie bialgebroid:

First, the direct sum $A \oplus A^* \to M$ has a Courant algebroid structure (Liu-Weinstein-Xu).

# In particular...



Take a Lie bialgebroid $(A, A^*)$ over a manifold $M$. Then we have two notions of *double* of the Lie bialgebroid:

First, the direct sum $A \oplus A^* \to M$ has a Courant algebroid structure (Liu-Weinstein-Xu).

# In particular...

Second, the cotangent double of $A$ (and of $A^*$)

$$\begin{array}{ccc} T^*A \simeq T^*A^* & \longrightarrow & A \\ \downarrow & & \downarrow \\ A^* & \longrightarrow & M \end{array}$$

has the structure of a double Lie algebroid. (Mackenzie)

# The two double of a Lie bialgebroid.

How are these two “doubles” related??

# The two double of a Lie bialgebroid.



$$\begin{array}{ccc} T(A \oplus A^*) & \longrightarrow & TM \\ \downarrow & & \downarrow \\ A \oplus A^* & \longrightarrow & M \end{array}$$

is the VB-Courant algebroid that is equivalent to the double Lie algebroid $T^*A \simeq T^*A^*$.

It is the tangent prolongation of $A \oplus A^* \to M$.

# The two double of a Lie bialgebroid.



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# Thank you for your attention!