Topological Phases and Topological Insulators

Manuel Asorey

Universidad de Zaragoza

IFWGP 2015

Zaragoza, September 2015

Physics and Topology

G. Gamow

Biography of Physics:
The great physicists from Galileo to Einstein

The future of physics:

In working up toward a dramatic conclusion of this volume

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- Einstein:
  Gravity $\Rightarrow$ Riemannian geometry
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Only number theory and topology still remain as pure mathematical disciplines without any physical application. Could it be that they will be called to help in our understanding of the riddles of nature?.

Current Physics

- Number Theory

– Integrable systems (Potts models)
– Casimir effect (Riemann zeta function)
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- Topology

– Dirac Monopoles
– Aharonov-Bohm effect
– Chern-Simons theory (knot theory)
– Topological Insulators

Topological Insulators

Zhang, Haldane, Kane

Topological Insulators

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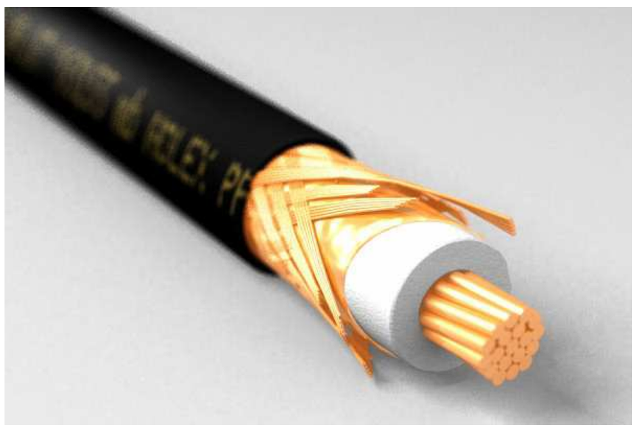
Topological insulators: New materials
Insulators in the bulk, but conductors on the boundary



Topological Phases

Particle in Magnetic Field in $\mathbb{R}^3$

$$L = \frac{1}{2} m \dot{\mathbf{x}}^2 + e\,\mathbf{A}.\dot{\mathbf{x}}$$

m mass of the particle
e electric charge

$\mathbf{A}$ magnetic vector potential $\mathbf{B} = \nabla \times \mathbf{A}$

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- metric independent
- constrained system $\mathbf{p} = e\,\mathbf{A}$

Topological Phases

Canonical formalism $(T^*\mathbb{R}^3, \omega_0)$

$$H = \frac{1}{2m}\,(\mathbf{p} - e\,\mathbf{A})^2$$

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Non-canonical transformation: $(T^*\mathbb{R}^3, \omega_0) \Rightarrow (T^*\mathbb{R}^3, \omega)$

$$\mathbf{p} \to \mathbf{p}' = \mathbf{p} - e\,\mathbf{A}$$

$$\omega_0 \Rightarrow \omega = \omega_0 + e\,\pi_0^*\,d\mathbf{A} = \omega_0 + e\,\pi_0^*\,\mathbf{F}$$

$$H \Rightarrow H' = \frac{1}{2m}\,\mathbf{p}'^2$$

Topological Phases

Constraints analysis in the topological phase reduce to a contact phase ($p' = 0$)

$$(T^*\mathbb{R}^3, \omega) \Rightarrow (\mathbb{R}^3, e\,\mathbf{F})$$

and

$$H' = 0$$

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Singular limit:

$$\lim_{m \to 0} \frac{1}{2m}\,\mathbf{p}'^2 \Rightarrow H' = 0$$

Topological Phases

Generalization for arbitrary Riemannian manifolds (M,g)

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Singular limit:

$$\lim_{m \to 0} \frac{1}{2m} (\mathbf{p}', \mathbf{p}')_g \Rightarrow H = H' = 0$$

Quantization

If M is even dimensional and F is regular $(\mathbf{M}, e\,\mathbf{F})$ is a symplectic manifold and no further reductions are needed

Quantization requires that

$$\left[\frac{e}{2\pi}\right]\mathbf{F} \in H^2(\mathbf{M}, \mathbb{Z})$$

Non-trivial topologies induce quantization of magnetic fluxes

Quantum states are sections of a line bundle $E(M, \mathbb{C})$ with a connection A such that $\pi^*\mathbf{F} = dA$ is the curvature of A by $\pi : E \to M$.

Quantization

If M is an oriented Riemannian manifold the quantum Hamiltonian is

$$\mathbb{H} = -\frac{1}{2m} d_A^* d_A = -\frac{1}{2m} \Delta_A, \tag{1}$$

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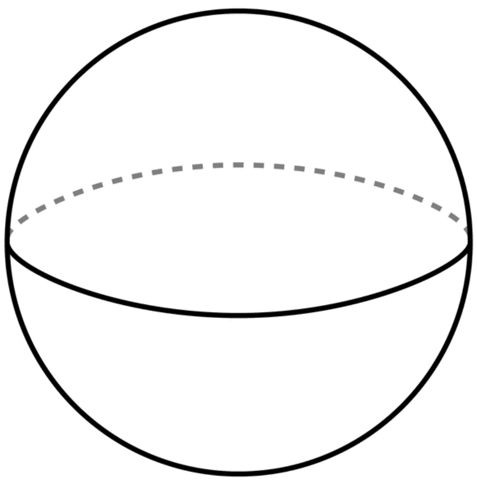
S^2 Sphere and Magnetic Monopole



$$\frac{e}{2\pi} \int_{S^2} \mathbf{F} = k \in \mathbb{Z}$$

$$\mathbf{B} = g \frac{\mathbf{x}}{||\mathbf{x}||^3} \qquad k = 2ge$$

Quantization

In complex coordinates $\mathsf{S}^2 = \mathbb{CP}^1$

$$\begin{aligned} z &= a e^{i\varphi} \tan \vartheta/2 \\ \bar{z} &= a e^{-i\varphi} \tan \vartheta/2 \end{aligned}$$

the Hamiltonian is

$$\mathbb{H} = -\frac{1}{2m}\left[\left(1 + \frac{z\bar{z}}{a^2}\right)^2 \partial\bar{\partial} + \frac{k}{2a^2}\left(1 + \frac{z\bar{z}}{a^2}\right)(z\partial - \bar{z}\bar{\partial}) - \frac{k^2}{4a^4} z\bar{z}\right]$$

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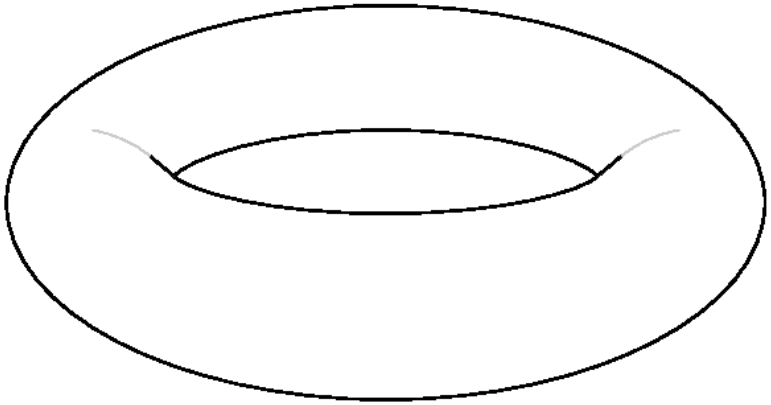
Energy levels (degeneracy $2l + |k| + 1$)

$$E_l = \tfrac{1}{2ma^2}\left[|k|(l+\tfrac{1}{2}) + l(l+1)\right] \qquad l = 0, 1, 2, \ldots$$

Eigenfunctions

$$\psi_j^l(z,\bar{z}) = \left(1+\frac{z\bar{z}}{a^2}\right)^{-k/2} z^j P_l^{(j,|k|-j)}\left(\frac{a^2 - z\bar{z}}{a^2 + z\bar{z}}\right) \begin{array}{l} l = 0, 1, 2, \ldots \\ j = -l, -l+1, \ldots, l+|k| \end{array}$$

Hall Effect in 2D Torus $\mathbb{T}$



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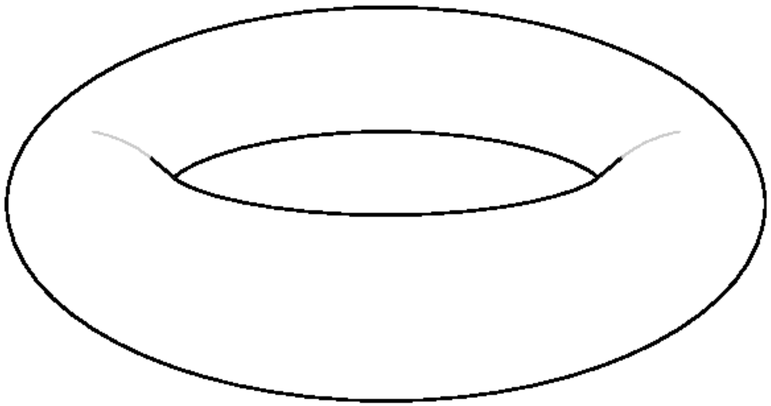


$$\frac{e}{2\pi}\int_T \mathbf{F} = k \in \mathbb{Z} \quad k = eB/2\pi$$

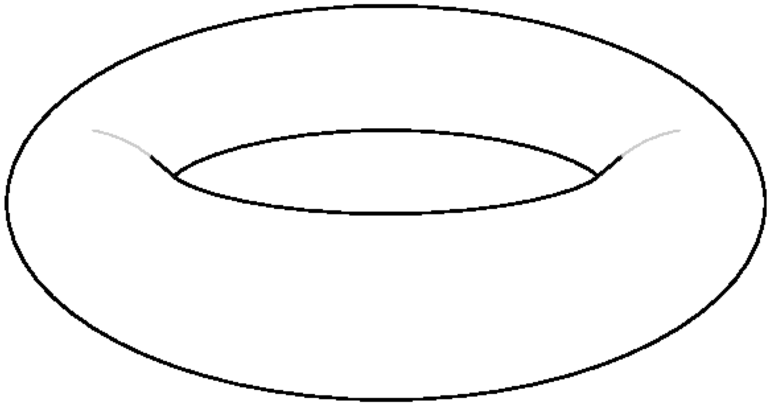
In complex coordinates: $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$

$$\mathbb{H} = -\frac{1}{2m}\left[4\partial\bar{\partial} + eB(z\partial - \bar{z}\bar{\partial}) - \frac{\mathrm{e}^2B^2}{4}z\bar{z}\right]$$

Energy levels (degeneracy: $|k|$)

$$E_n = \frac{2\pi|k|}{m}\left(n + \frac{1}{2}\right)$$

Hall Effect in 2D Torus $\mathbb{T}$



$$\frac{e}{2\pi}\int_T \mathbf{F} = k \in \mathbb{Z} \quad k = eB/2\pi$$

Ground State Eigenfunctions (degeneracy: $|k|$) :

Holomorphic sections of $E(T^2, \mathbb{C})$

$$\begin{aligned} \psi_j(z,\bar{z}) &= e^{k\pi z(z+\bar{z})/2}\,\Theta\begin{bmatrix} j/|k| \\ 0 \end{bmatrix}(|k|z, i|k|) \\ &= e^{k\pi z^2/2} \sum_{l\in\mathbb{Z}+j/|k|} e^{-\pi|k|l^2+i2\pi|k|lz} \\ j &= 0,1,2\ldots,|k|-1. \end{aligned}$$

Topological Phases

The massless limit $m \to 0$ can be analysed in two (equivalent) ways

- First constrain and then quantize
- First quantize and then constrain

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First constrain:

Constraints: $\mathbf{p}' = 0$

Reduced symplectic space: $(\mathbb{T}, e\,\mathbf{F})$

Hamiltonian: $H' = 0$

Topological Phases

Then quantize:

Topological Phases

Then quantize:

Classical and Quantum Anomalies
in the Quantum Hall Effect

M. Asorey[†]

Departamento de Física Teórica

Universidad de Zaragoza. 50009 Zaragoza. Spain

Topological Phases

Then quantize:

Journal of Geometry and Physics 11 (1993) 63–94
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Topological phases of quantum theories.
Chern–Simons theory

M. Asorey

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain [1]

Topological Phases

Then quantize:

- Prequantization condition

$$\frac{e}{2\pi}\int_T \mathbf{F} = k$$

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Then quantize:

- Prequantization condition

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- Holomorphic quantization:
  Quantum states $\approx$ holomorphic sections of a line bundle $E(T^2, \mathbb{C})$ with Chern class number $c_1(E) = k$

$$\mathcal{H}_k^0 = \{\xi : T^2 \to E; \xi \text{ is holomorphic}\}$$

Topological Phases

Riemann-Roch theorem

$$\dim \mathcal{H}^0_k = \frac{1}{8\pi}\int_\Sigma \sqrt{g}R + \frac{1}{2\pi}\int_\Sigma F + \dim \mathcal{H}^0_{2g-2-|k|}$$

- S^2 sphere $\dim \mathcal{H}^0_k = |k| + 1$
- $\mathbb{T}$ torus $\dim \mathcal{H}^0_k = |k|$
- Σ_g Riemann surface of genus g
  $\dim \mathcal{H}^0_k = |k| - g + 1 + \dim \mathcal{H}^0_{2g-2-|k|}$
  if $|k| - 2g + 2 > 0$ $\dim \mathcal{H}^0_{2g-2-|k|} = 0$
  (Kodaira's vanishing theorem)

Topological Phases

First quantize:

$$\mathbb{H} = -\frac{1}{2m} d_A^* d_A = -\frac{1}{2m} \Delta_A, \tag{3}$$

and then constrain:

The Hilbert space reduces to ground states $\dim \mathcal{H}_k^0$

- S^2 sphere $\dim \mathcal{H}_k^0 = |k| + 1$
- $\mathbb{T}$ torus $\dim \mathcal{H}_k^0 = |k|$
- Σ_g Riemann surface of genus g
  $\dim \mathcal{H}_k^0 = |k| - g + 1 + \dim \mathcal{H}_{2g-2-|k|}^0$

Semiclasical arguments

quantum states ⇔ volume of phase space

Semiclasical arguments

quantum states ⇔ volume of phase space
+ topological correction

Band structure

Periodic perturbation:

$$H'' = H' + V_0 \sin k\,\varphi_1 \sin k\,\varphi_2$$

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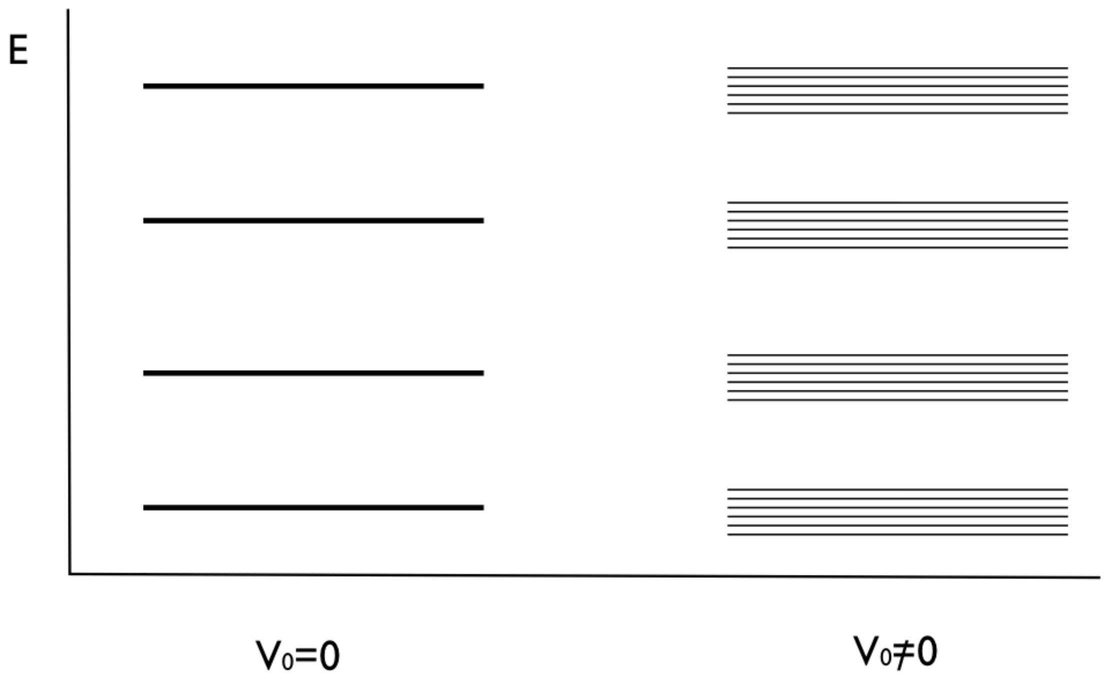
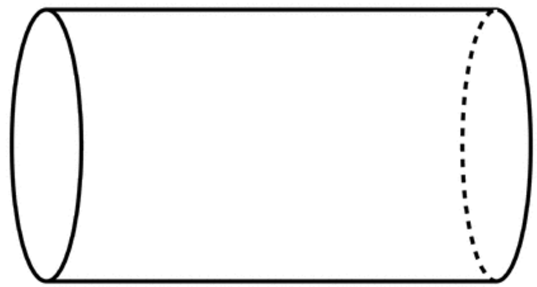


Hall effect with boundaries



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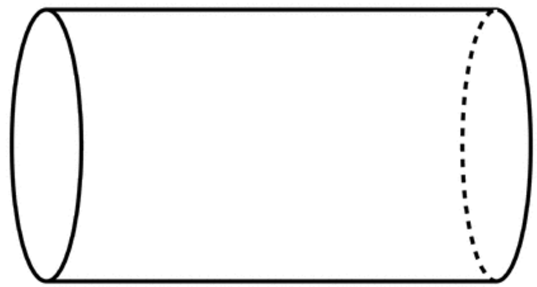
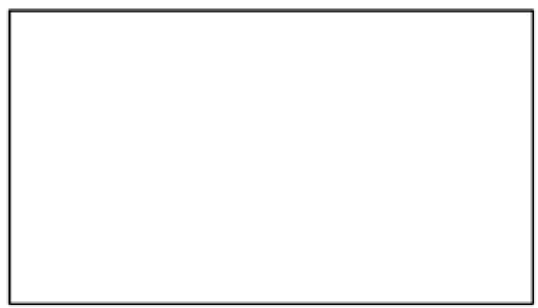
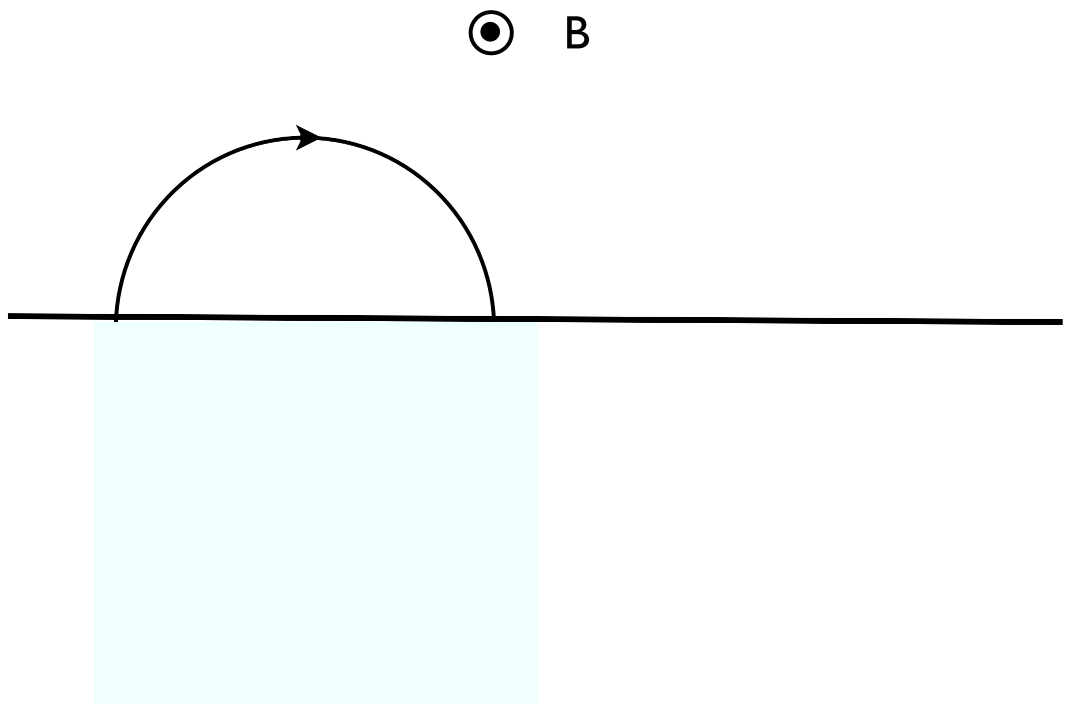




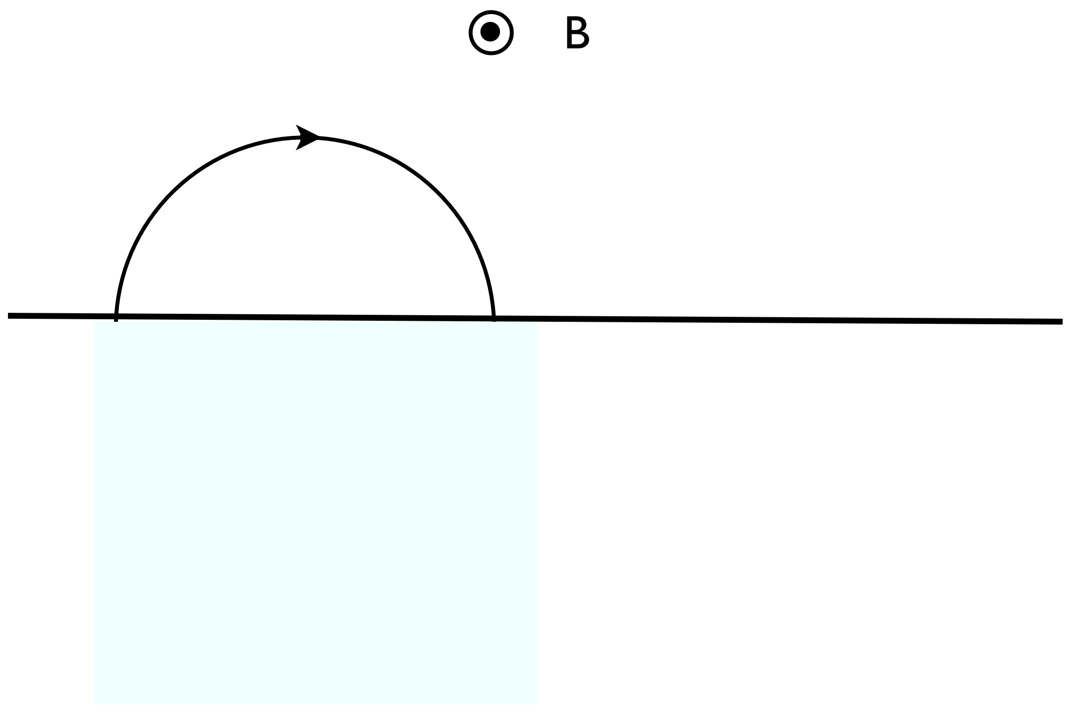
Hall effect with boundaries

Boundary effects:



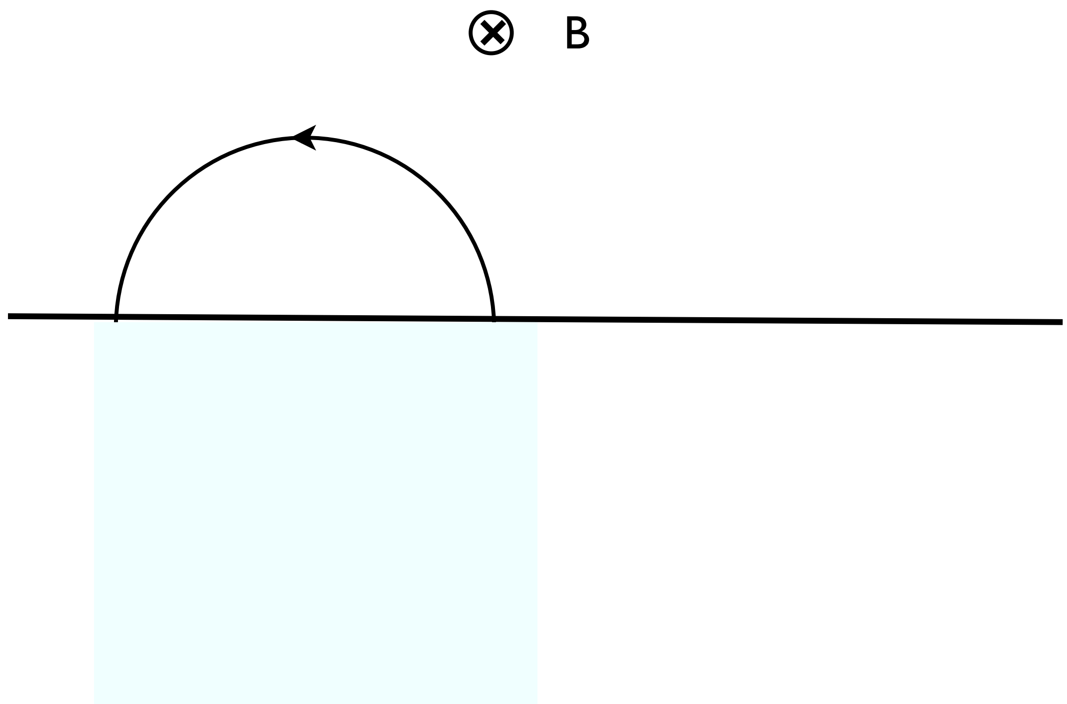
Hall effect with boundaries

Boundary effects: Time reversal symmetry?



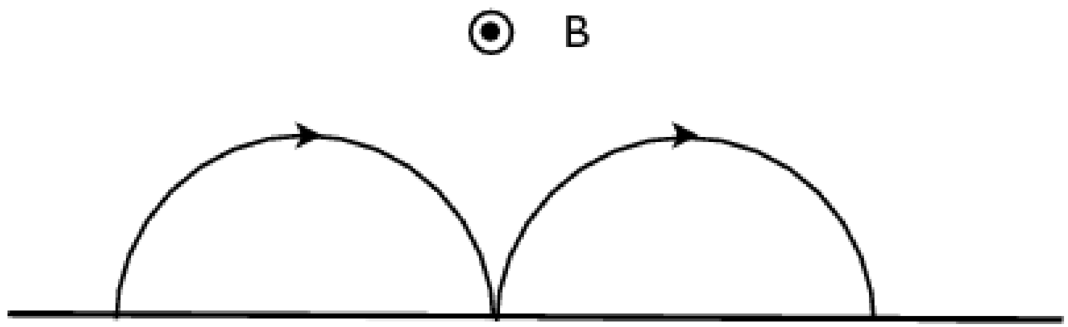
Hall effect with boundaries

Boundary effects: Time reversal symmetry



Hall effect with boundaries

Boundary effects: Time reversal symmetry breaking



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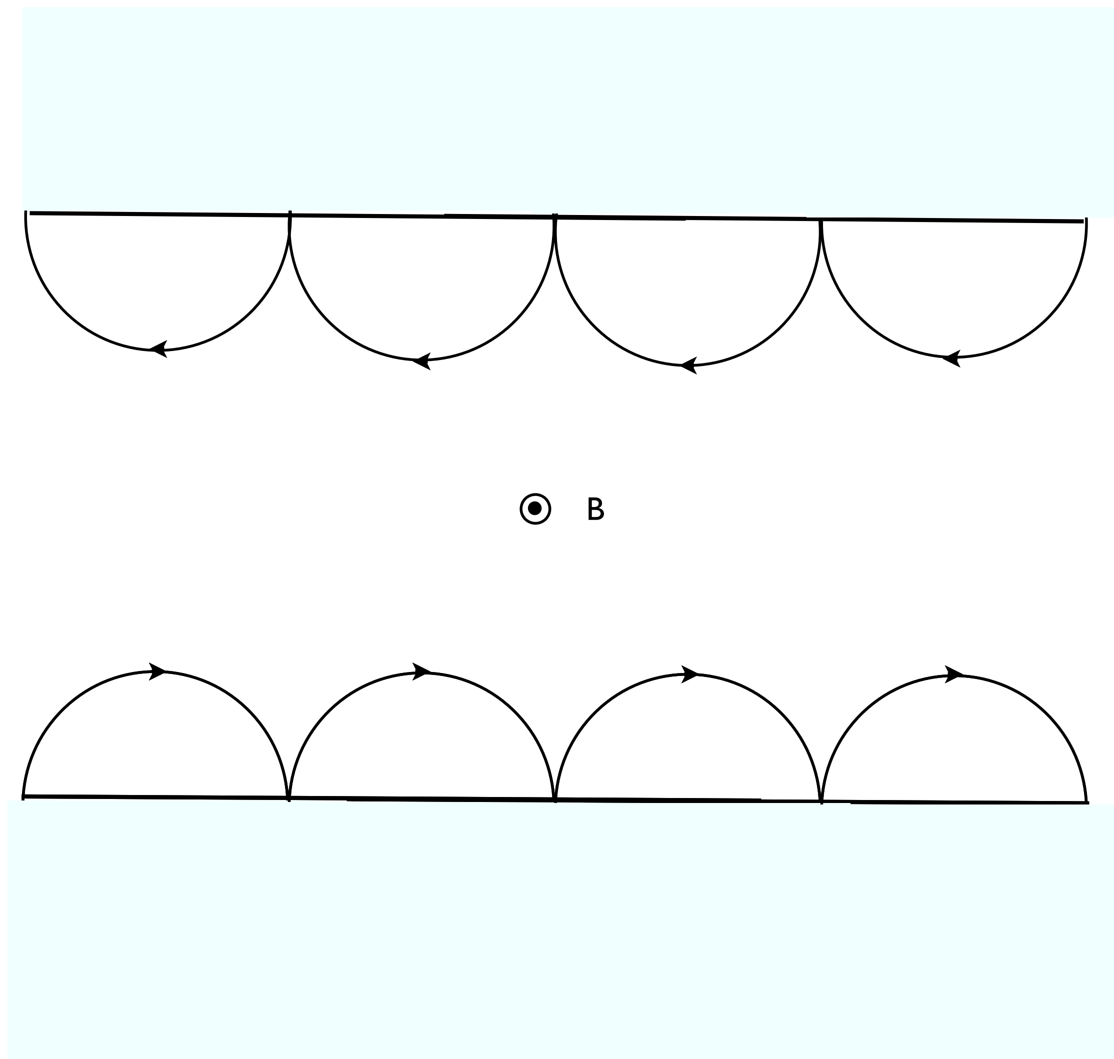
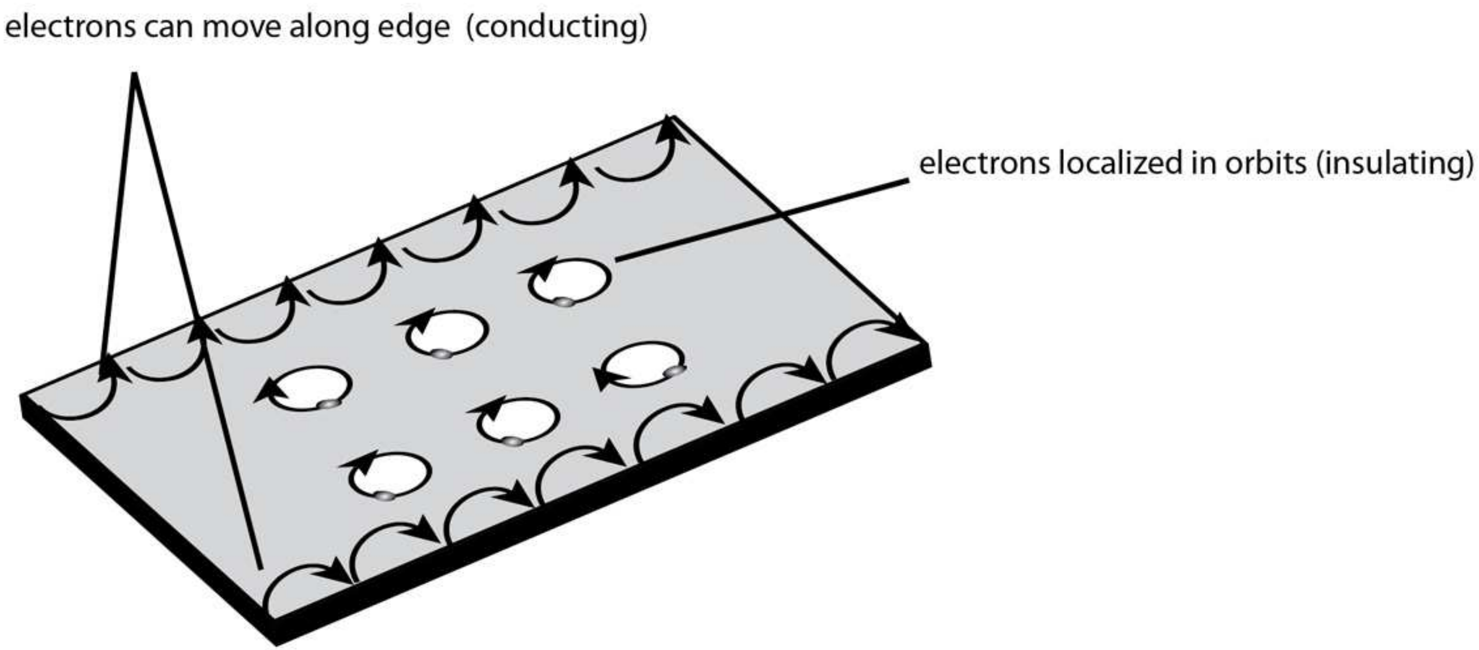
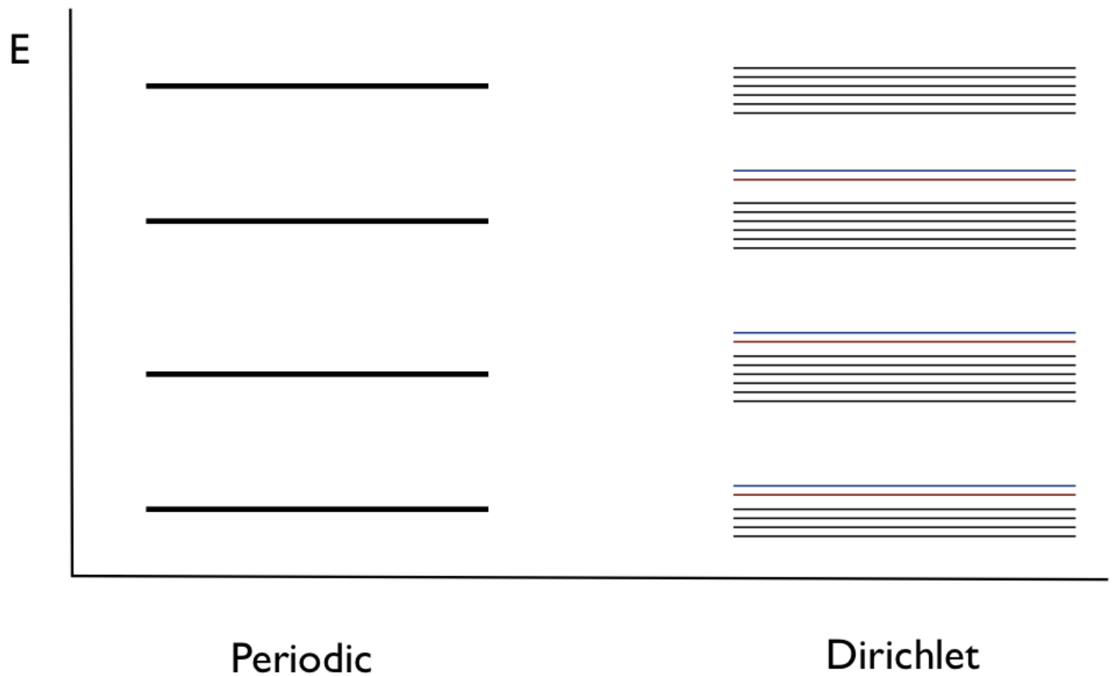


Hall effect with boundaries

Semiclassical argument :



Finite size effects



M. A., A. P. Balachandran, J.M. Perez-Prado, JHEP 2013 and [arXiv:1505.03461]

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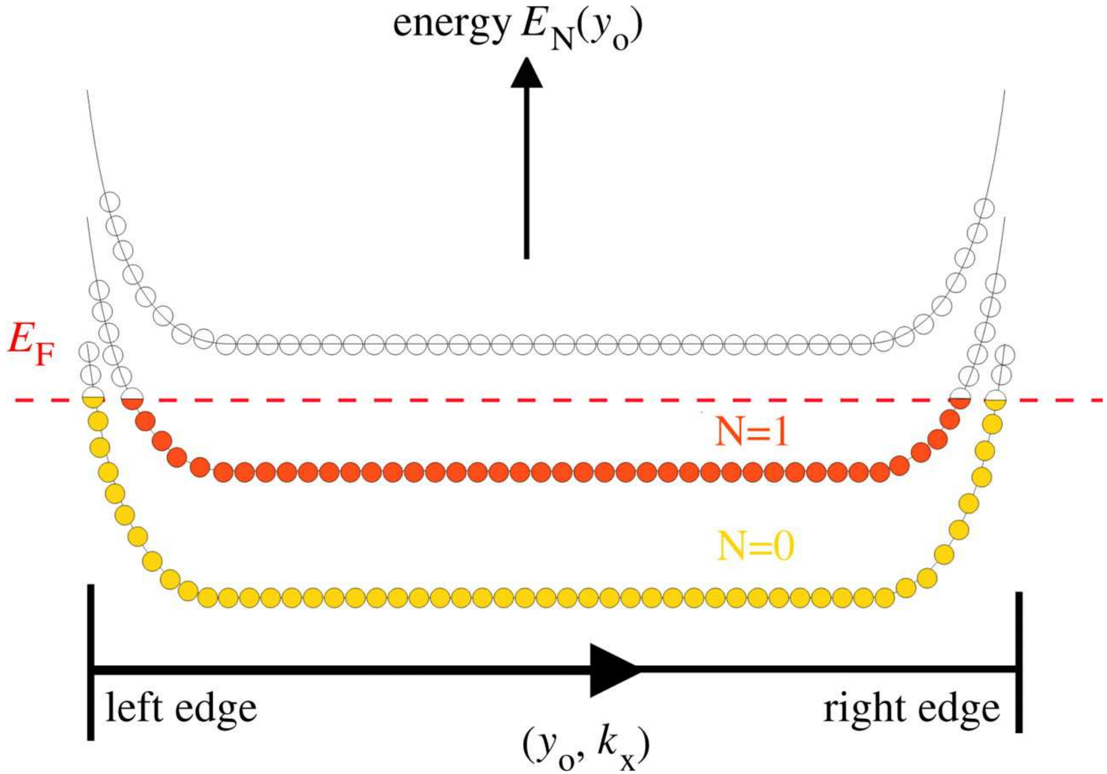
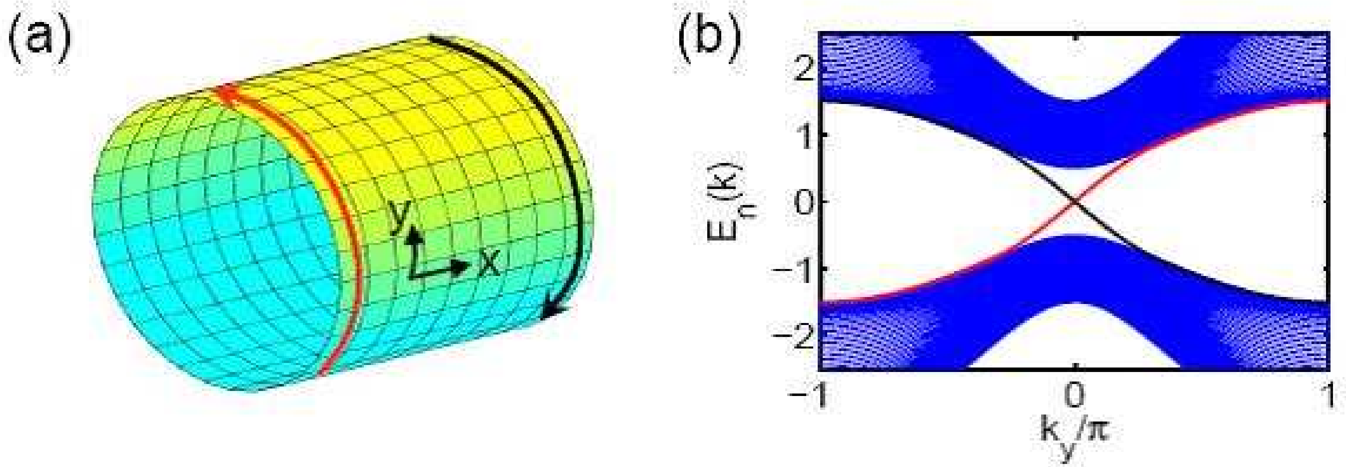


Band structure in Brillouin zone

$$\psi(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_k(r)$$



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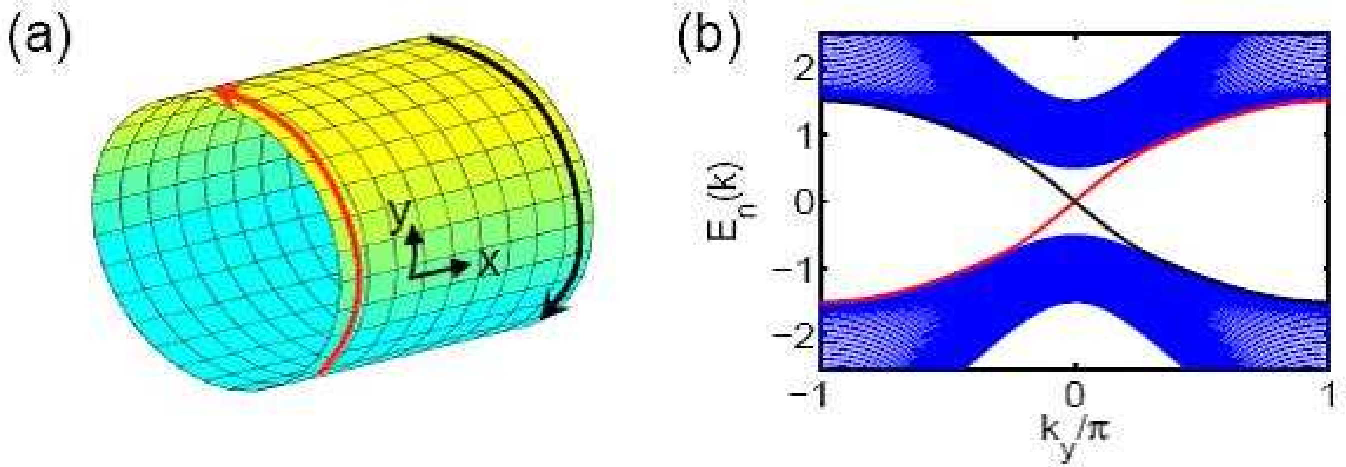
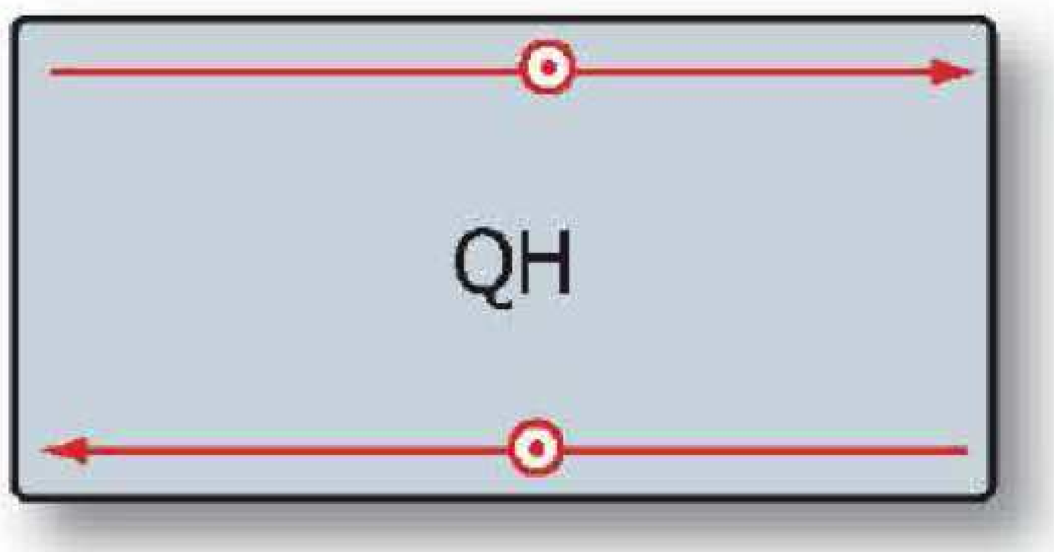




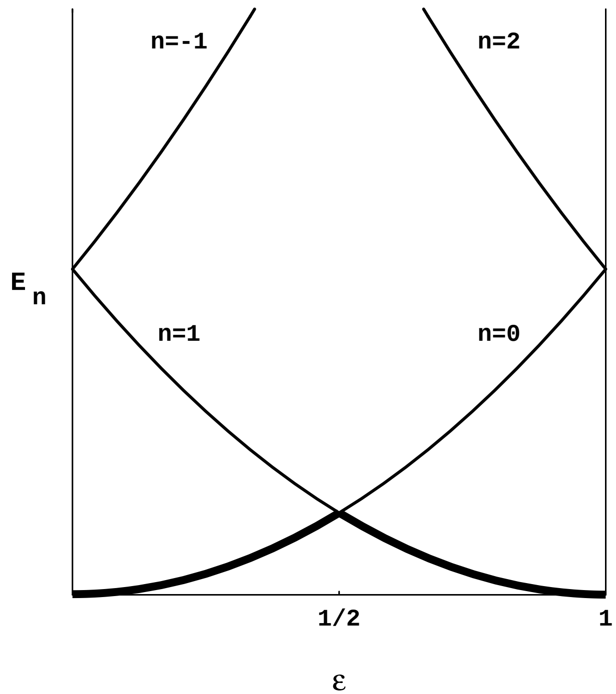
Boundary Insulators

Boundary interactions $\Rightarrow$ Anderson localization

Magnetic flux dependence

$$\mathbb{H} = -\frac{1}{2m}\left(\partial_\vartheta - i\frac{e\varphi}{2\pi}\right)^2 \qquad E_n = \frac{1}{2m}(n-\varepsilon)^2$$

$$\varepsilon = e\varphi/2\pi$$

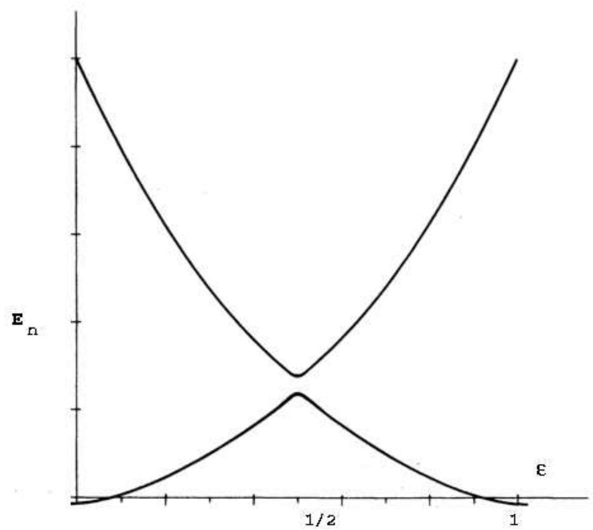


Time reversal invariance at $\varepsilon = 0, \frac{1}{2}$

Magnetic flux dependence

$$\mathbb{H} = -\frac{1}{2m}\left(\partial_\vartheta - i\frac{e\varphi}{2\pi}\right)^2 + V_0(1-\cos\vartheta)$$

$\varepsilon = e\varphi/2\pi$



degeneracy is not robust

Time Reversal and Kramers degeneracy

$s = \frac{1}{2}$ spin systems

$$\Theta\psi = e^{i\pi S_y}\psi^*$$

$$\Theta^2 = -I$$

Kramers theorem:

For a time reversal invariant Hamiltonian all energy levels are double degenerated

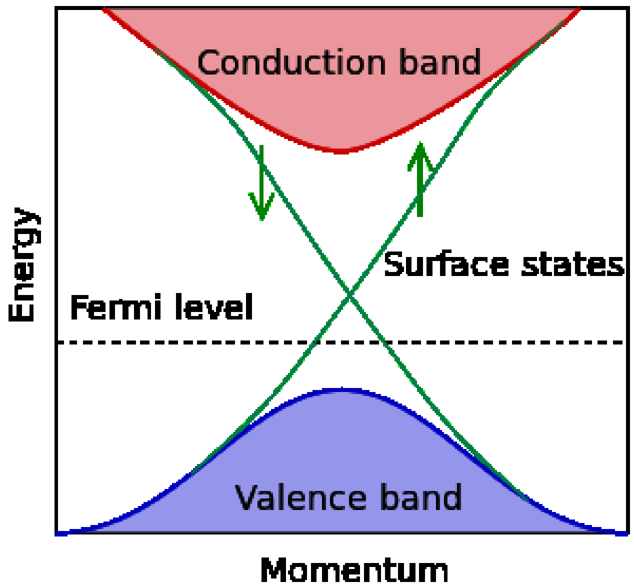
$$\Theta\psi = \lambda\psi, \qquad \Theta^2\psi = |\lambda|^2\psi = -\psi$$

For a non-degenerate energy level ψ

Quantum Spin Hall

Spin-Orbit interaction is TR invariant

$$H_I = g\mathbf{L} \cdot \mathbf{S}$$



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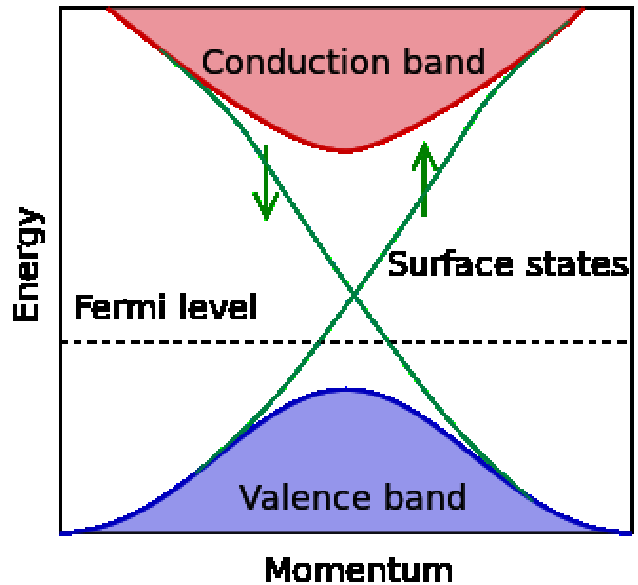
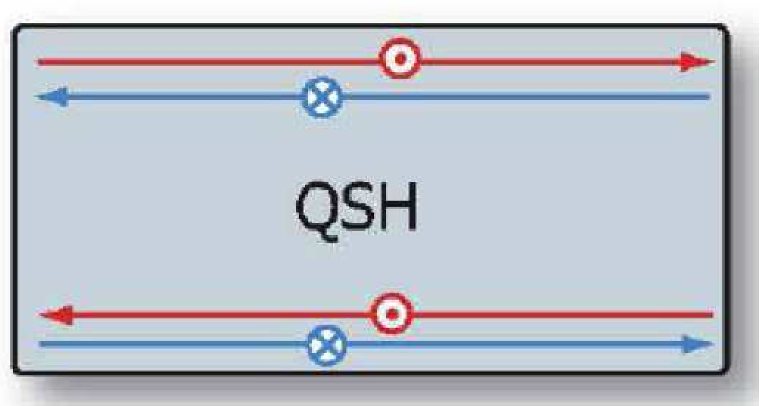




Time Reversal protection

Time reversal invariant interactions H_I

$$\Theta|k,\uparrow> = |-k,\downarrow>,\ \Theta|-k,\downarrow> = -|k,\uparrow>$$

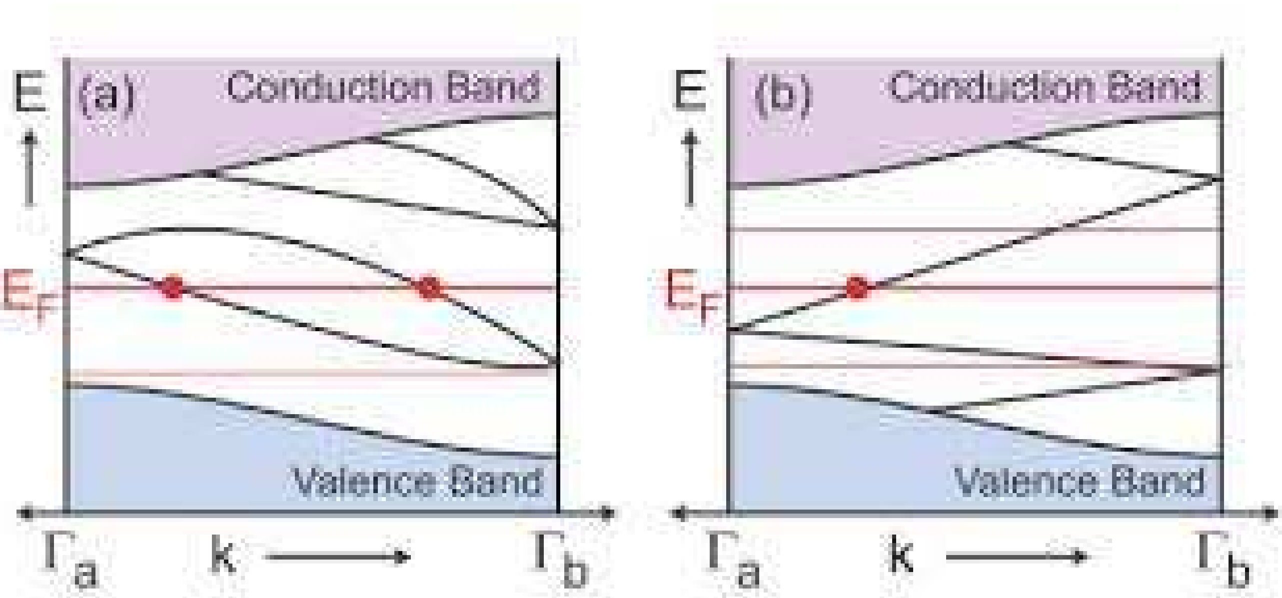
$$H_I\Theta = \Theta H_I$$

do no mix Kramers doublet states

$$< k,\uparrow|H_I|-k,\downarrow> =< k,\uparrow|H_I\Theta|k,\uparrow>$$

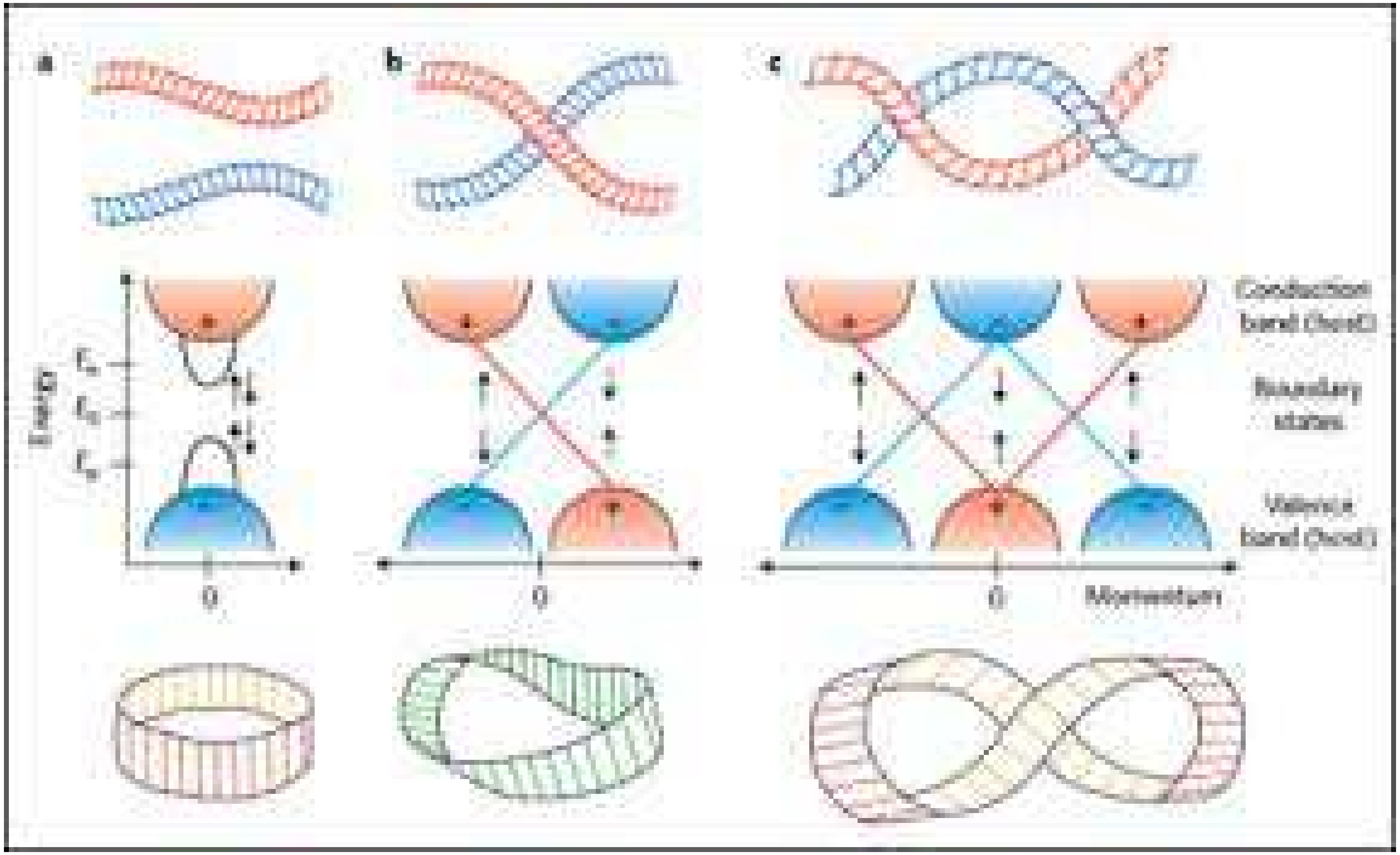
$$=< k,\uparrow|\Theta H_I|k,\uparrow> = -< -k,\downarrow|H_I|k,\uparrow> = 0$$

Topological Insulators: $\mathbb{Z}_2$ Index



normal insulator topological insulator

Topological Insulators



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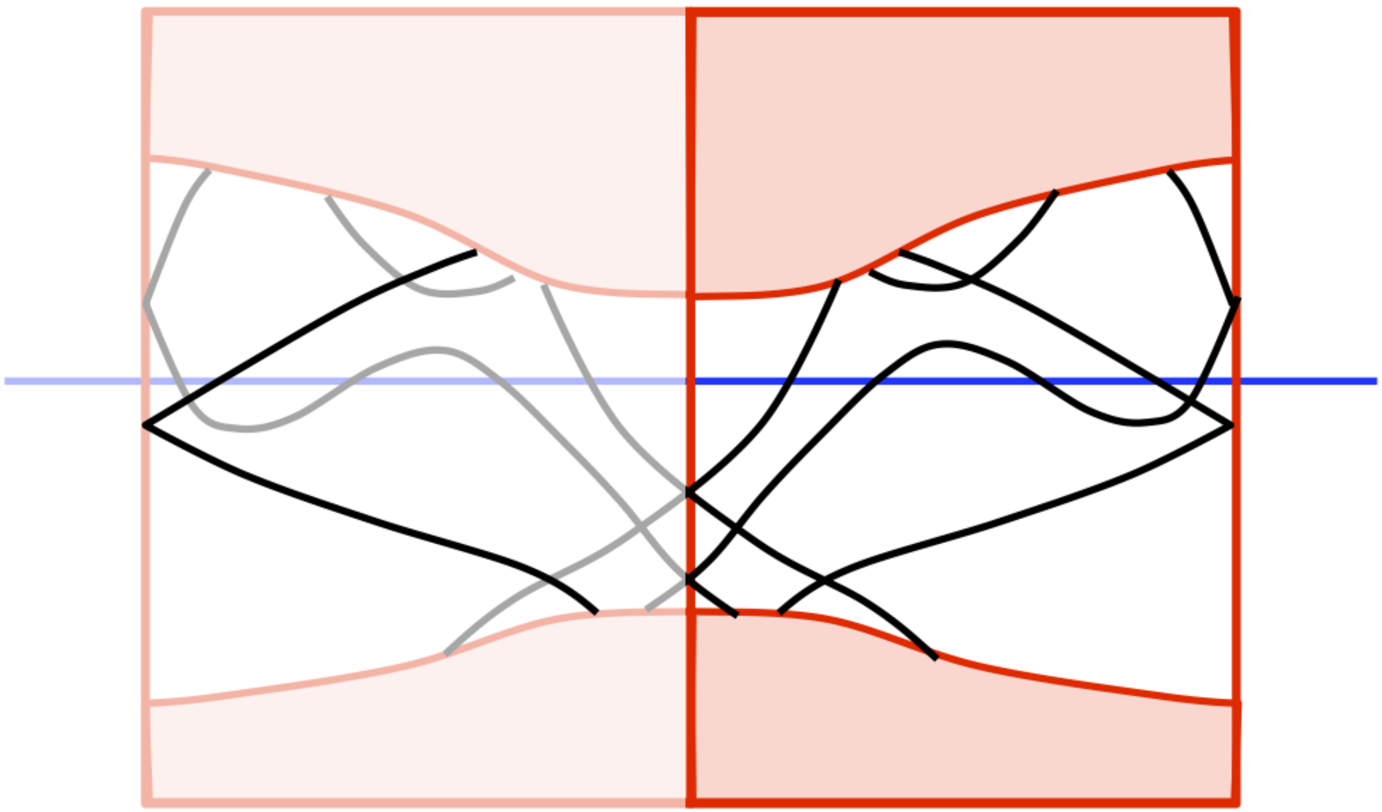


$$Z_2 = N_{\text{cross}}\ (\text{mod } 2) = \text{Invariant}$$

$\mathbb{Z}_2$ Index

Time reversal matrix

$$w_{mn}(k) = < u_m(k)|\Theta|u_n(-k) > \qquad |u_n(k)>$$ filled states

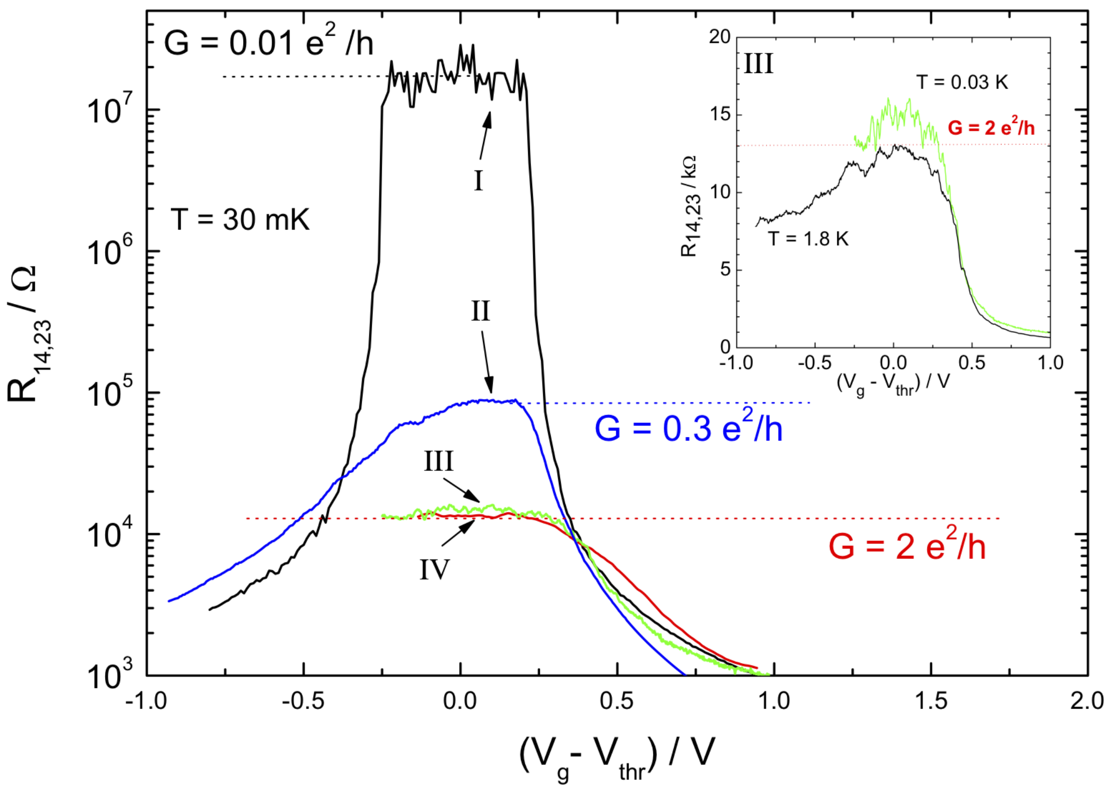
$$w_{mn}(k) = -w_{nm}(-k)$$

For TR invariant k_a the matrix $w(k_a)$ is antisymmetric

$\mathbb{Z}_2$ invariant ν is defined by

$$(-1)^\nu = \prod_a \frac{Pf(w(k_a))}{\det w(k_a)} = \pm 1$$

Topological Insulators



M. Koenig et al. Science (2007)

TOPOLOGICAL INSULATORS

- 2D topological insulators discovered in 2007

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